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High-accuracy statistical simulation of planetary accretion: I. Test of the accuracy by comparison with the solution to the stochastic coagulation equation

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The object of this series of studies is to develop a highly accurate statistical code for describing the planetary accumulation process. In the present paper, as a first step, we check the validity of the method proposed by Wetherill and Stewart (1989) by comparing the results obtained by their method with the analytical solution to the stochastic coagulation equation (or to a well-evaluated numerical solution). As the collisional probability A_{ij} between bodies with masses of im_1 and jm_1 (m_1 being the unit mass), we consider the two cases: one is $A_{ij} \propto i \times j$ and another is $A_{ij} \propto \min(i, j)(i^{1/3} + j^{1/3})(i + j)$. In both cases, it is known that runaway growth occurs. The latter case corresponds to a simplified model of the planetesimal accumulation. We assumed that a collision of two bodies leads to their coalescence. Wetherill and Stewart's method contains some parameters controlling the practical numerical computation. Among these, two parameters are important: the mass division parameter δ, which determines the mass ratio of the adjacent mass batches, and the time division parameter ϵ, which controls the size of a time step in numerical integration. Through a number of numerical simulations for the case of $A_{ij} = i \times j$, we find that when $\delta \leq 1.6$ and $\epsilon \leq 0.03$ the numerical simulation can reproduce the analytical solution within a certain level of accuracy independently of the size of the body system. For the case of the planetesimal accumulation, it is shown that the simulation with $\delta \leq 1.3$ and $\epsilon \leq 0.04$ can describe precisely runaway growth. Because the accumulation process is stochastic, in order to obtain reliable mean values it is necessary to take the ensemble mean of the numerical results obtained with different random number generators. It is also found that the number of simulations, N_c, demanded to obtain the reliable mean value is about 500 and does not strongly depend on the functional form of A_{ij}. From the viewpoint of the numerical handling, the above value of $\delta(\leq 1.3)$ and $N_c(\sim 500)$ are reasonable and, hence, we conclude that the numerical method proposed by Wetherill and Stewart is a valid and useful method for describing the planetary accumulation process. The real planetary accumulation process is more complex since it is coupled with the velocity evolution of the planetesimals. In the subsequent paper, we will complete the high-accuracy statistical code which simulate the accumulation process coupled with the velocity evolution and test the accuracy of the code by comparing with the results of N-body simulation.

1. Introduction

The numerical simulations of the planetary accumulation process have been so far investigated by two different ways: the N-body simulation and the statistical approach based on the Smoluchowski equation. In the N-body simulation (e.g., Lecar and Aarseth, 1986; Beaugé and Aarseth, 1990; Aarseth *et al.*, 1993; Kokubo and Ida, 1996) orbits of bodies are directly integrated by taking into account coalescences between them. Because of the limitation of computational ability, the number of the bodies treated in this technique is limited to only about 10000 bodies at most. However, in the early stage of the planetesimal accumulation, the number of planetesimals is inferred to be of the order of 10^{10} to 10^{12} (Greenberg *et al.*, 1978; Hayashi *et al.*, 1985). Even in the late stage of planetary accumulation, destructive collisions between planetesimals would create a large number of fragments. Thus, it is clear that N-body simulation cannot cover the whole process of the planetary accumulation though it is a powerful method under some suitable conditions.

In the stage where a huge number of bodies exist, it is necessary to introduce a statistical approach of some kind. In a statistical approach (e.g., Safronov, 1969; Greenberg *et al.*, 1978; Nakagawa *et al.*, 1983; Ohtsuki *et al.*, 1988), we usually describe the body system by the Smoluchowski equation which is expressed as (Smoluchowski, 1916)

$$\frac{dN_k}{dt} = \frac{1}{2}\sum_{i+j=k} A_{ij} N_i N_j - N_k \sum_{i=1}^{\infty} A_{ik} N_i, \qquad (1)$$

where N_k is the number (or the number density) of bodies with mass km_1 (m_1 being a unit mass and, from now on, put to be unity) and A_{ij} is the collisional probability between bodies with masses i and j. In a planetesimal accumulation process, the collisional probability A_{ij} is the function of the relative random velocity between the planetesimals with mass i and j and, thus, the velocity evolution of planetesimals must be calculated simultaneously with the accumulation process. Since the statistical approach is based on a number of assumption (e.g., spatially homogeneous distribution of planetesimals), the validity of the statistical approach is not clear. Hence, the validity of the statistical approach

should be investigated, for example, by the comparison with N-body simulations.

Furthermore, though the Smoluchowski equation seems to be rational, it is known that the Smoluchowski equation breaks down in cases of runaway growth (e.g., for a collisional probability $A_{ij} = i \times j$). In runaway growth, relatively large bodies grow faster than other smaller bodies because of strong positive dependence of the collisional probability on masses of bodies (Greenberg *et al.*, 1978; Wetherill and Stewart, 1989; Barge and Pellat, 1991; Spaute *et al.*, 1991; Kokubo and Ida, 1996). Such rapidly growing bodies in the high mass end of the mass distribution are called runaway bodies. Even among the runaway bodies, larger bodies grow faster than smaller ones and finally, one or a few bodies grow prominently, i.e., runaway growth of a few bodies occurs. Such a mode of the accumulation process cannot be intrinsically described by the Smoluchowski equation because it stands on the assumption that there exist a sufficiently large number of bodies in each relevant mass range. Hence, it is necessary to introduce a new statistical approach if we try to describe precisely runaway growth.

A more basic equation, which is valid in all types of the accumulation processes, was derived by Marcus (1968) in the field of atmospheric sciences. In his formulation, the accumulation process is regarded as a Markov process and a set of the body numbers in all mass bins $\{n_i\}$ is considered as a random variable where n_i is the number of bodies with mass i. His equation is the time development equation of the probability with which the body system is in a state $\{n_i\}$. Bayewitz *et al.* (1974) and Lushnikov (1978) solved this equation analytically in the cases of $A_{ij} = 1$ and $A_{ij} = i + j$, $i \times j$, respectively. On the other hand, in the field of planetary sciences, Tanaka and Nakazawa (1993) independently derived the same equation (they called it the stochastic coagulation equation) and found analytical solutions in the cases of $A_{ij} = 1, i + j$, and $i \times j$. Tanaka and Nakazawa (1994) also showed that, for the three cases of A_{ij} mentioned above, the Smoluchowski equation can approximately describe the number of bodies in each mass bin as long as the body mass is much smaller than a certain critical mass which depends on the functional form of A_{ij}. Although the stochastic coagulation equation gives an exact picture of the accumulation process, it is hopelessly difficult to solve this equation for other general cases of A_{ij} even by numerical procedures because it is quite difficult to cover all possible states $\{n_i\}$ perfectly. Thus, it is unsuitable to apply this equation to study of the real accumulation process.

Wetherill and Stewart (1989) developed an alternative numerical approach (this numerical method will be called WS89 method in the present paper) which supplements the defects of the Smoluchowski equation. This method were presented in more detail in Wetherill (1990). In the Smoluchowski equation the number of bodies N_i becomes a fractional value though the number of bodies should be, of course, an integer. To avoid this defect, in WS89 method the number of collisions during each numerical time step is forcely assigned to be an integer by the Monte Carlo method when the number of collisions is a fractional value and is smaller than 2×10^9 (Wetherill and Stewart, 1993). This method has two merits. One is that WS89 method guarantees the mass conservation in any cases of A_{ij} while the Smoluchowski equation violates it in some cases (e.g., $A_{ij} = i \times j$). Another is that it does not take long computing time to calculate evolution of the mass distribution compared with the case where we solve the same problem by means of the stochastic coagulation equation. In order to check whether WS89 method can succeed in describing runaway growth or not, Wetherill (1990) compared the numerical result with the analytical solution to the Smoluchowski equation for the case of $A_{ij} = i \times j$ where runaway growth occurs and found a fairly good agreement between them. However, such a comparison would be inadequate because the Smoluchowski equation cannot describe runaway growth of a few bodies mentioned above. The numerical results should be compared with the solutions to the stochastic coagulation equation.

Our principal aim in the present paper is to make clear the validity and usefulness of WS89 method by comparing the numerical results with the analytical solutions to the stochastic coagulation equation. To do so, we first reconstruct the algorithm according to Wetherill (1990) and confirm that our computational algorithm can follow the accumulation process within a sufficient accuracy for the cases where runaway growth does not occur. Next, we simulate the coagulation process for the case of $A_{ij} = i \times j$ (as well as the case of the planetesimal accumulation). As a result, we show that WS89 method can reproduce runaway growth precisely. In the subsequent paper, we will complete the statistical code to calculate growth of planetesimals coupled with their velocity evolution. By comparing the results obtained by our statistical code with those of N-body simulation, we will test the accuracy of our statistical code. In Section 2, we describe the stochastic coagulation equation and its analytical solution for the case of $A_{ij} = i \times j$. The numerical code is explained in Section 3. As will be mentioned in Section 3, WS89 method contains two technical parameters to be assigned: one is the mass division parameter, δ, which is the ratio of masses of adjacent mass batches and another is the time division parameter, ϵ, which determines the time interval used in the numerical integration. In Section 4, we compare the numerical solution obtained by WS89 method with the analytical solution to the stochastic coagulation equation for the case of $A_{ij} = i \times j$ and show that WS89 method can reproduce runaway growth precisely if the values of δ and ϵ are chosen appropriately. In Section 5, we repeat the similar comparison (as in the case of $A_{ij} = i \times j$) for the case of a simplified model of the planetesimal accumulation process and find the appropriate ranges of δ and ϵ with which the numerical simulation can give a precise picture of the planetesimal accumulation. The summary of the results obtained in the present study are shown in Section 6.

2. Stochastic Coagulation Equation and Its Solution

As a preparation of the later sections we review briefly the stochastic coagulation equation and its analytical solutions according to Tanaka and Nakazawa (1993). We consider a system which consists of a finite number of bodies with the total mass N. Since the unit mass m_1 is put to be unity, a body mass is given by an integer from 1 to N. A state of the mass distribution of the body system is expressed by

an N-dimensional vector $\mathbf{n} = (n_1, \cdots, n_N)$, called the state vector, where n_i is the number of bodies with mass i. Let a function $f(\mathbf{n}; t)$ be a probability that the system is in a state $\mathbf{n}$ at time t. The time development of $f(\mathbf{n}; t)$ is described by the following equation:

$$\begin{aligned}\frac{\partial}{\partial t} f(\mathbf{n}; t) = & \sum_{i,j=1, i\neq j}^{N} \frac{1}{2} A_{ij}(n_i + 1)(n_j + 1) \\ & \cdot f(\cdots, n_i + 1, \cdots, n_j + 1, \cdots, \\ & \qquad n_{i+j} - 1, \cdots; t) \\ & + \sum_{i=1}^{N} \frac{1}{2} A_{ii}(n_i + 2)(n_i + 1) \\ & \cdot f(\cdots, n_i + 2, \cdots, n_{2i} - 1, \cdots; t) \\ & - \sum_{i,j=1, i\neq j}^{N} \frac{1}{2} A_{ij} n_i n_j f(\mathbf{n}; t) \\ & - \sum_{i=1}^{N} \frac{1}{2} A_{ii} n_i (n_i - 1) f(\mathbf{n}; t).\end{aligned} \tag{2}$$

The first two terms of the right hand side of Eq. (2) represent the probabilities of transitions from other states into a state $\mathbf{n}$ by the coalescence between bodies with masses i and j and between bodies with the same mass i. The remaining two terms represent the probabilities of transitions from the state $\mathbf{n}$ to other states. An expectation of the random variable n_k is given by

$$\langle n_k \rangle \equiv \sum_{\mathbf{n}} n_k f(\mathbf{n}; t), \tag{3}$$

which corresponds to N_k for the case of the Smoluchowski equation. Multiplying both sides of Eq. (2) by n_k and taking a sum over all states, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\langle n_k \rangle = & \frac{1}{2} \sum_{i,j=1}^{N} A_{ij} \delta_{k,i+j} \langle n_i (n_j - \delta_{ij}) \rangle \\ & - \sum_{i=1}^{N} A_{ik} \langle n_k (n_i - \delta_{ik}) \rangle,\end{aligned} \tag{4}$$

where the bracket $\langle\ \rangle$ denotes the same meaning as that of Eq. (3) and δ_{ij} is the Kronecker delta. If $\langle n_i (n_j - \delta_{ij}) \rangle$ can be replaced by $\langle n_i \rangle \langle n_j \rangle$, this equation coincides with the Smoluchowski equation (1) (Lushnikov, 1978; Tanaka and Nakazawa, 1993).

Analytical solutions to the stochastic coagulation equation have been obtained for the special three cases of A_{ij}: $A_{ij} = 1$, $i + j$, and $i \times j$ (Bayewitz *et al.*, 1974; Lushnikov, 1978; Tanaka and Nakazawa, 1993). For the later convenience, we present here the analytical solution for the case of $A_{ij} = i \times j$:

$$\langle n_k \rangle = {}_N C_k e^{-k(N-k)t} f_k(t), \tag{5}$$

where ${}_N C_k$ is the binomial coefficients. Furthermore, $f_k(t)$ is a function which can be obtained by solving successively the following equation,

$$\frac{\partial f_k(t)}{\partial t} = \frac{1}{2} \sum_{i,j=1}^{N} ij \delta_{i+j,k}\, {}_k C_i e^{-ijt} f_i(t) f_j(t) \tag{6}$$

under the initial condition of $f_i(0) = \delta_{i1}$ (note that all bodies are assumed to have a unit mass initially, i.e., at $t = 0$). By the detailed analysis of solution (5), it is known that only the largest body grows prominently after $t = 1/N$, i.e., runaway growth of one body begins to occurs.

In the present study we pay especially attention to the growth of the runaway body (i.e., the largest body) as well as the second largest body because we are interested in precise description of runaway growth. According to Tanaka and Nakazawa (1994), the expectation values of the masses of the largest body, M_1, and the second largest body, M_2, are approximately given by

$$M_1 = \sum_{k=k_1}^{N} k\langle n_k \rangle \qquad \text{and} \qquad M_2 = \sum_{k=k_2}^{k_1-1} k\langle n_k \rangle, \tag{7}$$

where masses k_1 and k_2 are defined, respectively, by

$$\sum_{k=k_1}^{N} \langle n_k \rangle = 1 \qquad \text{and} \qquad \sum_{k=k_2}^{N} \langle n_k \rangle = 2. \tag{8}$$

Using Eqs. (5), (7), and (8), Tanaka and Nakazawa (1994) evaluated M_1 and M_2 (for the case of $N = 500$) as a function of the normalized time η defined by

$$\eta = Nt, \tag{9}$$

where η is the time normalized by the mean coalescent time at the initial stage. They also showed that the time developments of M_1 and M_2 become almost independent of N if the mass is normalized by $N^{2/3}$ and the time is measured by the renormalized time u defined by

$$u = \frac{N^{2/3}}{\sqrt{2(N^{2/3} - 1)}} \left(\eta - 1 + \frac{1}{N^{2/3}} \right) \tag{10}$$

(Note that u is the time normalized by the characteristic time during which the mass of the largest body increases by a factor of e just when runaway growth starts). Thus, in order to see the detailed behaviors of M_1 and M_2 around the stage of $\eta = 1$, we will frequently use the $M/N^{2/3} - u$ diagram in the later section.

3. Preparation of the Numerical Code

We reconstruct the numerical code, which enables us to simulate the accumulation process, as closely as possible after WS89 code (Wetherill and Stewart, 1989, 1993; Wetherill, 1990). Let $N(m, \eta)$ be the mass distribution function of bodies at time η where η is the normalized time defined by Eq. (9). In order to calculate numerically the time variation of the mass distribution function $N(m, \eta)$, we divide the mass coordinate discretely into a number of batches in a logarithmic way:

$$m_{i+1} = \delta\, m_i, \qquad i = 1, \cdots, n_\mathrm{b} - 1 \tag{11}$$

where m_i is the representative mass of batch i and n_b is the total number of batches. Furthermore, δ (>1) is a constant of the order of unity which will be called the mass division parameter from now on. If the value of δ is too large, physically realistic description of the accumulation process might fail. If it is very close to 1, on the other hand, we are forced

to prepare a large number of numerical batches and spend long computing time though the simulation would describe the accumulation process precisely. Thus, we have to choose the value of δ appropriately in the numerical simulations.

Numerically, the mass distribution function is expressed by a set of the numbers of bodies contained within batch i, $N(m_i, \eta)$ (which, from now on, will be abbreviated as N_i). The growth of bodies belonging to batch i is treated in two ways. If the mass of the merged body, $m_i + m_j$, is smaller than $(m_i + m_{i+1})/2$, the body still belongs to batch i; the number of bodies in batch j is decreased by ν_{ij} and the number of bodies in batch i remains unchanged whereas the total mass in batch i is increased by $\nu_{ij} m_j$ and the total mass in batch j is decreased by $\nu_{ij} m_j$. In the above, ν_{ij} is the number of collisions during the (numerical) time interval $\Delta t (= \Delta\eta / N)$ between bodies belonging to a target batch i and a projectile batch j $(i \geq j)$ and is given by

$$\nu_{ij} = \frac{N_i (N_j - \delta_{ij})}{1 + \delta_{ij}} A_{ij} \Delta t. \tag{12}$$

On the other hand, when the combined mass, $m_i + m_j$, is greater than $(m_i + m_{i+1})/2$, the merged body should be placed to a higher batch (say, batch k); the numbers of bodies in both batches i and j are decreased by ν_{ij} and the total masses in batches i and j are decreased by $\nu_{ij} m_i$ and $\nu_{ij} m_j$, respectively. Furthermore, the number of bodies in batch k is increased by ν_{ij} and its mass is increased by $(m_i + m_j)\nu_{ij}$.

In the Smoluchowski equation, at least mathematically, N_i is not prevented from having an unphysical fractional value. Especially, for the case where N_i is relatively small, N_i needs to be an integer. In order to guarantee this, Wetherill and Stewart (1993) introduce the following device. When ν_{ij} is equal to or greater than 2×10^9*, we permit ν_{ij} has a fractional value. But, when ν_{ij} is smaller than 2×10^9, we reevaluate ν_{ij} so that ν_{ij} becomes an integer, that is,

$$\nu'_{ij} = [\nu_{ij}] + \beta, \tag{13}$$

where ν'_{ij} is the reevaluated value we use, $[\nu_{ij}]$ is integer part of ν_{ij} and β is given by

$$\beta = \begin{cases} 1, & \nu_{ij} - [\nu_{ij}] \geq \zeta, \\ 0, & \nu_{ij} - [\nu_{ij}] < \zeta. \end{cases} \tag{14}$$

In the above ζ is a random number between 0 and 1 generated by the random number generator.

In cases of runaway growth, spaces between batches becomes wide as the accumulation process proceeds. When the mass ratio between two batches is larger than a certain value, we create empty batches. Furthermore, another device is introduced to describe properly runaway growth. If the mass of the runaway body exceeds the half of the total mass, we regard the runaway body as an independent body and pick it off from the mass distribution. From Eq. (1), we can readily derive the growth equation of the independent body (i.e., the runaway body) expressed as

$$\frac{dM_r}{dt} = \sum_{i=1}^{n_b} i A_{iM_r} N_i, \tag{15}$$

where M_r is the mass of the runaway body.

Wetherill (1990) did not describe in detail how to choose a relevant size of a time step in the numerical simulations. In our study, the time step $\Delta\eta$ is determined by the following way. According to the general rule of numerical integrations, $\Delta\eta$ must be chosen so that during $\Delta\eta$ the change of the particle number ΔN_i is much smaller than the particle number itself, N_i, in any batches. Furthermore, the increment of the mass ΔM_r of the runaway body must be also much smaller than its mass M_r during $\Delta\eta$. Thus, in the present study, we choose the time step $\Delta\eta$ so as to satisfy the following condition:

$$\Delta\eta = \epsilon \times \min\left(\frac{N_i}{\dot{N}_i}, \frac{M_r}{\dot{M}_r}, 0.1 \right), \tag{16}$$

where dot denotes the derivative with respect to the normalized time η and ϵ is a parameter, called the time division parameter, to be assigned before the numerical simulation. Note that $\min(N_i/\dot{N}_i) < 0.1$ at $\eta \simeq 1$.

In order to check whether our code can reproduce WS89 code or not, we simulate the same problem under the same condition as that of Wetherill (1990), that is,

$$A_{ij} = i + j, \quad N(m_i, 0) = N\delta_{i1}, \quad \text{and} \quad N = 1 \times 10^{20}. \tag{17}$$

As for δ and ϵ, we put to be $\delta = 1.1$ and $\epsilon = 0.01$ (δ is the same value as that adapted by Wetherill (1990) but, as mentioned above, the value of ϵ is taken independently). For the case of $A_{ij} = i + j$, the analytical solution to the Smoluchowski equation is found as (Trubnikov, 1971)

$$N_k = N \frac{k^{k-1}}{k!} e^{-\eta} (1 - e^{-\eta})^{k-1} \exp[-k(1 - e^{-\eta})]. \tag{18}$$

Note that, in this case, comparison between our numerical solution and the analytical solution to the Smoluchowski equation is meaningful because the Smoluchowski equation

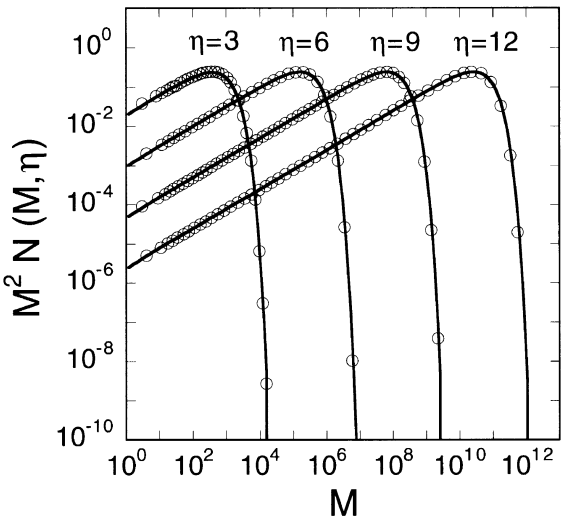

Fig. 1. Comparison between the analytical solution (curves) and the numerical solution (circles) for the case of $A_{ij} = i + j$. The mass distribution functions $N(M, \eta)$ multiplied by M^2 are shown for typical evolutionary times η. Parameters are put to be $\delta = 1.1$, $\epsilon = 0.01$, and $N = 1 \times 10^{20}$.

*the maximum integer expressed by 32-bit computer memory, i.e., 2^{31}.

gives the precise solution of the accumulation process for $A_{ij} = i + j$ as long as the mass of the largest body is much smaller than the total mass (Tanaka and Nakazawa, 1994). In Fig. 1, we illustrate the mass distribution functions obtained numerically at typical epochs as well as the corresponding analytical solutions. From this figure, we can see that the numerical simulation can reproduce the analytical solution and find the almost same accuracy of the numerical simulation as that of Wetherill (1990)**.

4. Comparison between Numerical and Analytical Solutions for the Case of $A_{ij} = i \times j$

4.1 Method of comparison

In order to check the validity of WS89 method, we perform numerical simulations using our numerical code for the case of $A_{ij} = i \times j$ where runaway growth inevitably occurs. The initial condition is taken as

$$N_k = N\delta_{1k}, \tag{19}$$

where N is the total number of bodies with unit mass. Analytical solution to the stochastic coagulation equation is already presented in Section 2. We also mentioned runaway growth of one body starts at $\eta \simeq 1$ (i.e., $t \simeq 1/N$).

We compare the time evolution of the masses of the largest and the second largest bodies, M_1 and M_2, obtained from the numerical simulation with those obtained from the solution to the stochastic coagulation equation. In the early stage (i.e., $\eta \ll 1$), there exist a large number of bodies in any relevant mass batches and, then, the Smoluchowski equation is still valid (Tanaka and Nakazawa, 1994). Wetherill (1990) showed that their numerical result agrees with the solution to the Smoluchowski equation. Thus, in the early stage, we can use WS89 method as long as we choose appropriately the technical parameters δ and ϵ. In runaway growth, as the accumulation process proceeds, the number of runaway bodies becomes a few or one. Generally, because of its statistical property, the Smoluchowski equation cannot describe the subsequent stage where a few bodies grow prominently. Thus, to check validity of WS89 method, we compare the result of WS89 method with the analytic solution to the stochastic coagulation equation in the case of $A_{ij} = i \times j$ which is the only case where the solution is known and runaway growth occurs. In order to check whether WS89 method reproduces the solution to the stochastic coagulation equation, the check should be done in the late stage, where the Smoluchowski equation breaks down. Note that the case of $A_{ij} = i \times j$ has a special property that, for a sufficiently small mass k, the solution $\langle n_k \rangle$ to the stochastic coagulation equation agrees with that to the Smoluchowski equation even in the late stage (Tanaka and Nakazawa, 1994). As runaway growth of one body proceeds, except for the largest one, there exist only sufficient small bodies of which numbers is well described by the Smoluchowski equation in the case of $A_{ij} = i \times j$. Then, by subtracting the total mass of the small bodies from the total mass in the system, we also obtain the precise value of the runaway body's mass from the Smoluchowski equation in this case. In this way, for $A_{ij} = i \times j$, the Smoluchowski equation also can describe the stage where runaway growth of one body proceeds sufficiently, though it cannot describe the stage where runaway growth of one body starts (i.e., around $\eta = 1$). Hence, we make a comparison between the results of WS89 method and the analytic solution to the stochastic coagulation equation especially in the stage where runaway growth of one body starts.

In the comparison, we pay attention to the time development of the masses of the largest and the second largest bodies, M_1 and M_2. For the stage where runaway growth starts, it is meaningless to compare the numerical result obtained from a single run with the analytical solution because M_1 and M_2 obtained from the stochastic coagulation equation are expectation values. In Fig. 2a, we show the time variation of M_1 (circles) and M_2 (triangles), which are obtained from a single run, for the case of $N = 1 \times 10^3$, $\delta = 1.3$, and $\epsilon = 0.01$ as a function of the renormalized time u defined by Eq. (10). In this figure two solutions are quite different from each other: both M_1 and M_2 obtained from the numerical simulation change discontinuously owing to the coalescence with other bodies whereas those obtained from the stochastic coagulation equation change smoothly since they are expectation values. Hence, the analytical solution should be compared with the ensemble mean obtained from a large number of independent runs with different random number generators. The ensemble mean of a body mass at time η (or u), M, is given by

$$M = \frac{1}{N_c} \sum_{i=1}^{N_c} M^i, \tag{20}$$

where N_c is the number of independent runs and M^i is the body mass at time η obtained by the i-th run. In Fig. 2b, we show the ensemble means of M_1 and M_2 obtained by performing 1000 runs (i.e., $N_c = 1000$). The ensemble means of M_1 and M_2 calculated from the numerical simulation agree with the analytical solutions to the stochastic coagulation equation even in the stage where runaway growth of one body starts. The comparison between the numerical and analytical solutions in general view is also shown in Fig. 2c. In both early and late stages, two solutions agree with each other. This means that WS89 method is able to describe precisely the accumulation process if we use suitable parameters δ and ϵ.

The technical parameters δ and ϵ govern essentially the degree of accuracy of the numerical solutions. If we can find their values with which the numerical simulation reproduces the analytical solution within a certain level of accuracy and if these values are suitable for actual computational runs in a sense of the needed memory size as well as the elapsed computing time, then we can say that WS89 method is valid and useful. Hence, the present aim is to find a suitable set of δ and ϵ. To find appropriate ranges of δ and ϵ, we make a comparison between two solutions in a quantitative manner. We pay our attention to the following five quantities, which characterize the time development of M_1 and M_2 (the five quantities are illustrated schematically in Fig. 3 in which the abscissa is measured by u to magnify the picture around $\eta = 1$). As for the largest body, two quantities are picked up: one is the

**Wetherill (1990) seems to mislabel values of the abscissa in Fig. 4. The values should be from 0 to 13, not from 2 to 15.

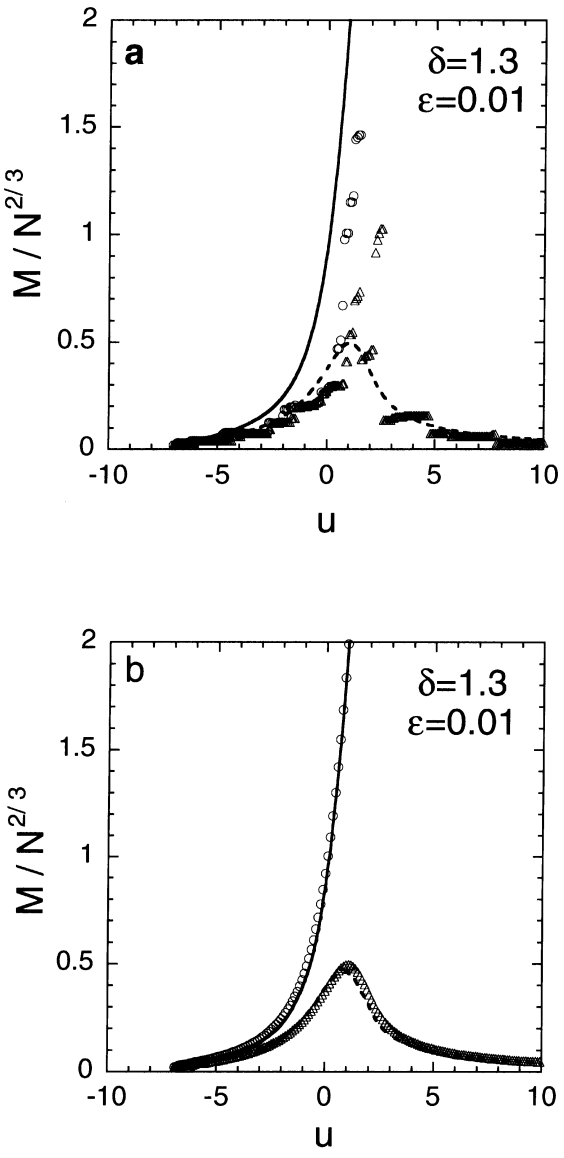

Fig. 2. The growth of the mass of the largest body M_1 (circles) as well as that of the second largest body M_2 (triangles) for the case of $A_{ij} = i \times j$ (parameters are put to be $\delta = 1.3, \epsilon = 0.01$, and $N = 1 \times 10^3$). Panel (a) shows M_1 and M_2 obtained by a single run while panels (b) and (c) show those evaluated as the ensemble means of 1000 independent runs. The vertical axis shows the mass normalized by $N^{2/3}$ and abscissa denotes the renormalized time u. In each panel, expectation values of M_1 and M_2 evaluated by the analytical solution are also shown by solid and dashed curves, respectively.

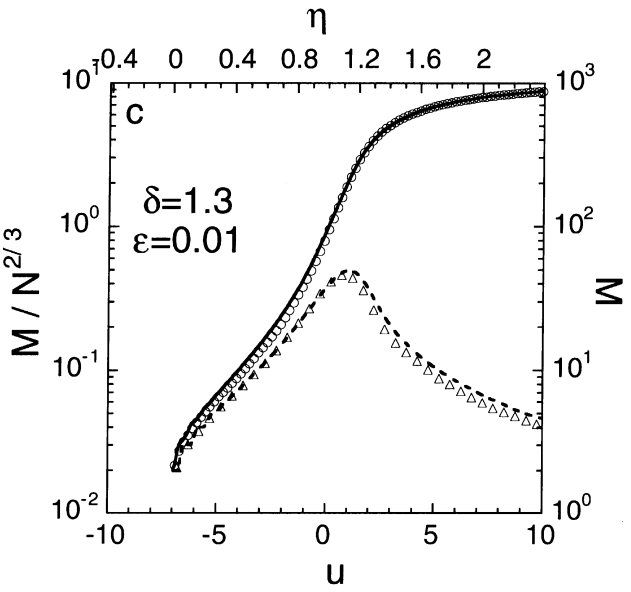

Fig. 2. (continued).

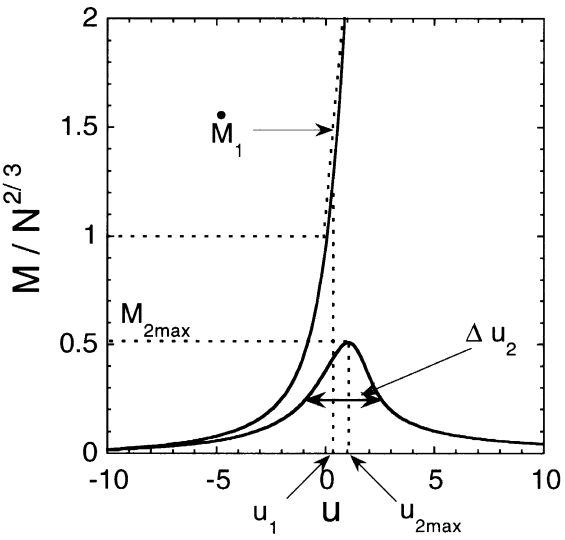

Fig. 3. Five quantities which are used for the comparison between the numerical and the analytical solutions in our present study (Note that as the abscissa we use u in place of η in order to magnify the picture around the stage of beginning of the runaway growth). In the figure, $\dot{M}_1$ is the average growth speed of the mass of the largest body and u_1 is the time when the mass of the largest body, M_1, becomes three times as large as that of the second largest body, M_2. Furthermore, $M_{2\max}$ and $u_{2\max}$ are the maximum value of M_2 and the instant when $M_2 = M_{2\max}$, respectively, and Δu_2 is the period, during which $M_2 \geq \frac{1}{2} M_{2\max}$. Note that η_1, $\eta_{2\max}$, and $\Delta\eta_2$ correspond to u_1, $u_{2\max}$, and Δu_2, respectively, through Eq. (10) or (31).

average growth speed, $\dot{M}_1$, which is defined by the reciprocal of the time interval between two epochs, $M/N^{2/3} = 1$ and 2. Another is the time, η_1 (corresponding to u_1), at which the mass of the largest body becomes three times as large as that of the second one; η_1 is introduced to see in detail the connection between masses of the largest and the second largest bodies. As for the growth of the second largest body, we consider three quantities, that is, the mass, $M_{2\max}$, the time, $\eta_{2\max}$ (corresponding to $u_{2\max}$), and the time interval, $\Delta\eta_2$ (corresponding to Δu_2). Here, $M_{2\max}$ is the maximum mass of the second largest body (note that, as shown schematically in Fig. 3, the mass of the second largest body initially increases and afterward tends to decrease), $\eta_{2\max}$ is the time at which the mass of the second largest body attains the maximum value, and $\Delta\eta_2$ is the time interval during which it stays with mass greater than $M_{2\max}/2$.

The degree of accuracy of these quantities is measured by the relative errors defined by $(X_n - X_a)/X_a$ where X_n and X_a

Table 1. Relative errors of the five quantities in typical two cases, $\delta = 1.3$ and 3.0. In both cases, N and ϵ are put to be 1000 and 0.01, respectively.

δ	$\dot{M}_1$	$M_{2\max}$	$\Delta\eta_2$	η_1	$\eta_{2\max}$
1.3	0.057	−0.066	−0.011	0.0070	−0.022
3.0	0.61	0.0037	−0.33	0.24	0.21

are the quantities obtained from the numerical and analytical solutions, respectively. As a level of accuracy, we demand the following condition:

$$\left|\frac{X_{\rm n} - X_{\rm a}}{X_{\rm a}}\right| \leq \xi \tag{21}$$

where

$$\xi = \begin{cases} 0.3, & \text{for} \quad \dot{M}_1,\, M_{2\max},\ \text{and}\ \Delta\eta_2, \\ 0.04, & \text{for} \quad \eta_1 \ \text{and}\ \eta_{2\max}. \end{cases} \tag{22}$$

Usually, the relative errors of η_1 and $\eta_{2\max}$ are small. Therefore, in Eq. (22), we demand a high level of accuracy for these relative errors.

4.2 Validity check of Wetherill and Stewart's method

According to the method of comparison mentioned in the last Subsection, we investigate in detail the validity of WS89 method by simulating the accumulation process for the case of $A_{ij} = i \times j$ where runaway growth inevitably occurs in the course of accumulation. We adopt $N_{\rm c} = 1000$, which will be justified later in Subsection 4.3. We first show the typical two examples of numerical simulations with $N = 1000$, i.e., the cases of $(\delta, \epsilon) = (1.3, 0.01)$ and $(3.0, 0.01)$. In the former case, the simulation satisfies the demanded level of accuracy mentioned above and the latter is the case where we cannot obtain precise numerical results. Numerical errors in the two cases are tabulated in Table 1. For the case of $\delta = 1.3$, the errors of all quantities are suppressed sufficiently under the level of accuracy demanded by inequality (21). For the case of $\delta = 3.0$, however, the errors are too large: especially, the relative errors of η_1 and $\eta_{2\max}$, are 0.24 and 0.21, respectively. Thus, we cannot accept the result of this simulation as a solution though the error of $M_{2\max}$ happens to be small. For the case of $\delta = 3$, the mass distribution is described by only 6 batches. Hence, it is hard to express the exact accumulation process by such a small number of batches. We see from these two examples that, as a reasonable result, the simulations with small δ could describe accurately the accumulation process.

In order to see in detail how the relative errors depend on the adapted value of δ, we have made 10 simulations with various values of δ (N and ϵ are fixed to be 1000 and 0.01, respectively). As illustrated in Fig. 4a, when δ is smaller than 2.9, the relative errors of $\dot{M}_1$ and $\Delta\eta_2$ are within the demanded level. When δ becomes large beyond 2.9, the relative errors increase suddenly and, at the same time, the results of the simulation do not satisfy the condition (21). On the other hand, the errors of $\eta_{2\max}$ and η_1 are at a low level as long as we are concerned with the case of $\delta \leq 1.5$. For $\delta \geq 1.5$, they begin to increase almost monotonously and rapidly with an increase in δ (see Fig. 4b). When δ exceeds

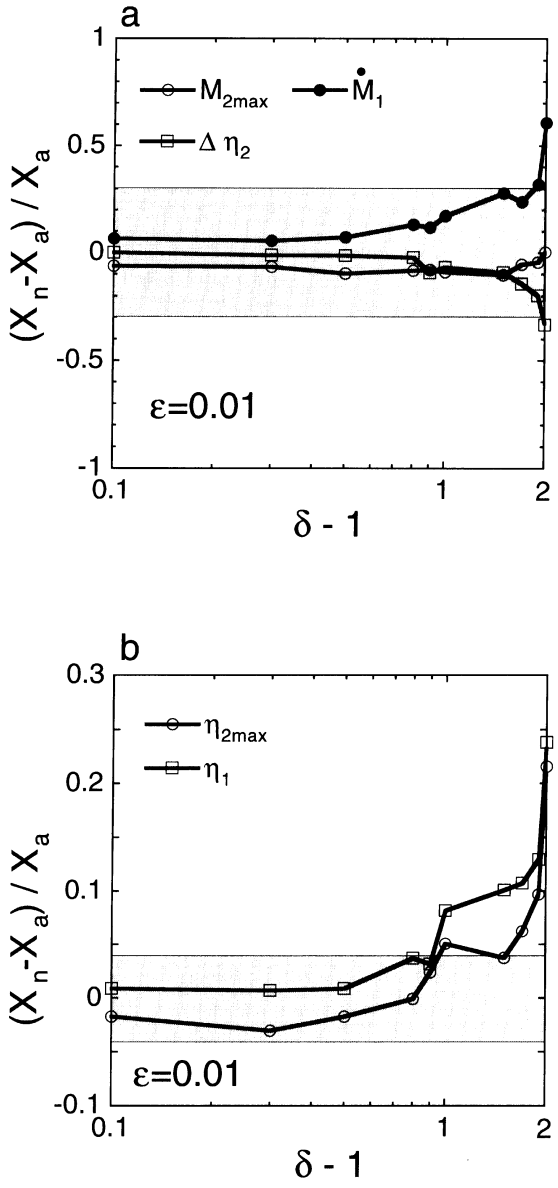

Fig. 4. The behaviors of the relative errors of $\dot{M}_1$, $M_{2\max}$, $\Delta\eta_2$ (panel (a)), η_1, and $\eta_{2\max}$ (panel (b)) against $\delta - 1$ for the case of $N = 1000$ (ϵ is fixed to be 0.01). The shaded region denotes the admissible range of errors demanded by condition (21).

1.9, the errors of $\eta_{2\max}$ and η_1 go out of the admissible range given by the inequality (21). Thus, it is ascertained that, for the case of $\epsilon = 0.01$, the numerical simulation is valid because it can reproduce approximately the accumulation process (but within the demanded level of accuracy) when δ is equal to or less than 1.9. Such a value of δ, beyond

which the simulation fails, is called the critical value of δ and denoted by δ_c. For the case of $\epsilon = 0.01$ and $N = 1000$, we have $\delta_c = 1.9$. In general, the value of δ_c is determined in Fig. 4b instead of Fig. 4a.

The accuracy of the numerical simulation depends not only on the mass division parameter δ but also on the time division parameter ϵ which determines the time increment $\Delta\eta$ (see Eq. (16)). As seen from Fig. 4, if we choose a very large value of δ, the numerical solution departs far from the analytic one even if ϵ is chosen appropriately. The same would be true for ϵ: for the value of ϵ larger than a certain critical value ϵ_c, the error of the numerical solution is beyond the demanded level of accuracy even if δ is properly chosen. The numerical solution would reproduce accurately the analytical one if both δ and ϵ are put to be sufficiently small. Hence, there exists a certain curve on the $\delta - \epsilon$ plane such that the error of the numerical solution is suppressed within the demanded level of accuracy as long as a set of δ and ϵ is inside the region enclosed by the critical curve (which from now on is called the $\delta_c - \epsilon_c$ curve). In order to find the $\delta_c - \epsilon_c$ curve, we have made numerical simulations for the various values of δ (from 1.1 to 2.0) and ϵ (from 0.01 to 1.0). The $\delta_c - \epsilon_c$ curve obtained from the simulations is shown in Fig. 5 for the case of $N = 1000$. From Fig. 5, we see that, if δ is smaller than 1.5, the maximum admissible value of ϵ (i.e., ϵ_c) is 0.1 and that ϵ_c decreases suddenly with an increase in δ when $\delta > 1.5$.

The $\delta_c - \epsilon_c$ curve depends on the total number of the system N. In order to see the dependence of the curve on N, we made two kinds of simulations: one is simulations to find δ_c as a function of N under the fixed value of ϵ ($= 0.01$) and another is those to find ϵ_c under the fixed value of δ ($= 1.5$). The results of the former and the latter simulations are shown in Figs. 6a and b, respectively. Both δ_c and ϵ_c decrease with an increase in N and tend to converge to certain values, $\delta_c = 1.6$ and $\epsilon_c = 0.03$. This convergence means that we can find an appropriate set of δ and ϵ irrespective of the size of the body system if it is large enough. In Fig. 5 we also show the $\delta_c - \epsilon_c$ curve in the limit of large N presumed from the above results. Though it is difficult to understand quantitatively the behaviors of ϵ_c and δ_c shown in Fig. 6, we say in a qualitative sense that their behaviors are natural: for the case of small N, a numerical simulation is accomplished with the relatively small number of time steps so that large values of δ and ϵ are admitted.

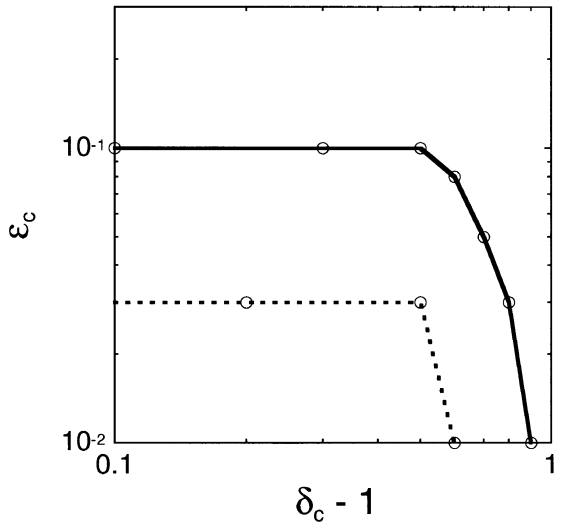

Fig. 5. The $\delta_c - \epsilon_c$ curve for the case of $N = 1000$ (solid curve). The numerical simulation with δ and ϵ, which are chosen from the region enclosed by this curve, could describe the accumulation process within the demanded level of accuracy as long as $N \leq 1 \times 10^3$. The same curve is also drawn for the limiting case of large N (dashed line).

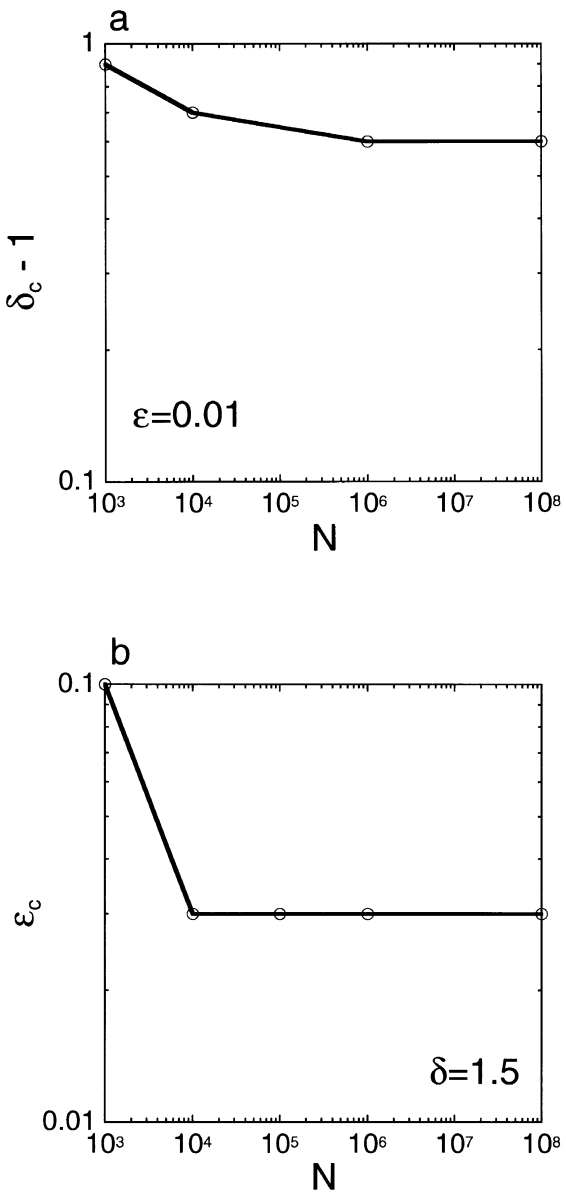

Fig. 6. The behaviors of δ_c (panel (a)) and ϵ_c (panel (b)) against the total body number of the system N. In panels (a) and (b), the other parameters are fixed to be $\epsilon = 0.01$ and $\delta = 1.5$, respectively.

4.3 Sample number for evaluating the ensemble mean

As seen in Fig. 2a, the numerical solution obtained by a single run is quite different from the analytic solution (or more exactly, the expectation value obtained from the solution to the stochastic coagulation equation) because of the large dispersion. In order to see the average behavior of the

particle growth by the numerical simulation, we take the ensemble mean, performing many runs with different random number generators. We examine here a suitable choice of the number of independent runs for evaluating the ensemble mean. For various numbers of the runs (N_c = 10, 100, 500, 1000, 2000, and 4000), we have calculated the ensemble means of the five quantities introduced in Subsection 4.1 and evaluated their relative errors (δ and ϵ are fixed as $\delta = 1.3$ and $\epsilon = 0.01$). Figure 7 shows the dependence of the relative errors of $\dot{M}_1$, $M_{2\max}$, and $\Delta\eta_2$ (Fig. 7a) and of η_1 and $\eta_{2\max}$ (Fig. 7b) as a function of the number of simulations N_c. From this figure, we can see that the relative errors converge if we average the quantities over 100 (or 1000) simulations for the case of $\dot{M}_1$, $M_{2\max}$, and $\Delta\eta_2$ (or η_1 and $\eta_{2\max}$). All of the numerical results in Subsection 4.2, are evaluated by the ensemble means averaged over 1000 independent runs, which is justified by the above result.

Strictly speaking, the number of samples N_c should be determined on the basis of the ratio of the dispersion to the mean value. As an example, we consider the averaged value of the mass of the largest body, M_1 (given by Eq. (20)), and its standard deviation, $M_{1,s}$, which is defined by

$$M_{1,s} = \sqrt{\frac{1}{N_c}\sum_{i=1}^{N_c}(M_1^i - M_1)^2}. \tag{23}$$

In Fig. 8, we show the ratio of the standard deviation to the averaged value evaluated from 1000 numerical simulations (i.e., $N_c = 1000$) in which N, δ, and ϵ are put to 1×10^6, 1.2, and 0.01, respectively. Just before the start of runaway growth (i.e., $M_1/N^{2/3}$ is about 0.1 and the renormalized time u is about -5 (see also Fig. 2a)), $M_{1,s}/M_1$ is relatively small. But in the vicinity of $M_1/N^{2/3} \simeq 1$ ($u \simeq 0$), the ratio rises almost to 0.6. Afterwards, the ratio decreases gradually and becomes smaller than 0.1 after $M_1/N^{2/3} > 7$ ($u = 3$). As conjectured from Fig. 2a, this behavior of $M_{1,s}/M_1$ means that the stochastic property becomes distinct around the stage of the beginning of runaway growth.

From the above results, we can evaluate the number of simulations N_c demanded for obtaining the reliable mean value M_1 (within an accuracy of, say, 3 percent) as

$$N_c = \left(\frac{1}{0.03}\frac{M_{1,s}}{M_1}\right)^2. \tag{24}$$

Equation (24) yields $N_c = 400$ and 11 when $M_1/N^{2/3} = 1$ and 7, respectively. Therefore, 1000 simulations used in Subsection 4.2 is sufficient. It is worthwhile to note that, as readily conjectured, the behavior of $M_{1,s}/M_1$ shown in Fig. 8

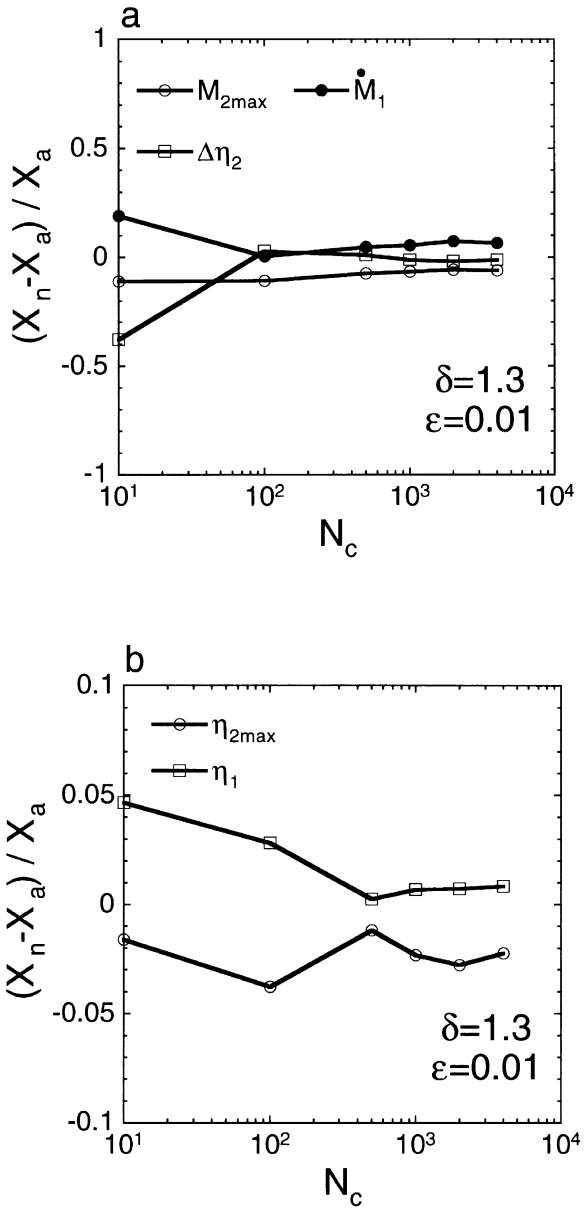

Fig. 7. The dependences of the relative errors of $\dot{M}_1$, $M_{2\max}$, $\Delta\eta_2$ (panel (a)), η_1, and $\eta_{2\max}$ (panel (b)) on the number of independent simulations, N_c. The other parameters are put to be $\delta = 1.3$, $\epsilon = 0.01$ and $N = 1 \times 10^3$.

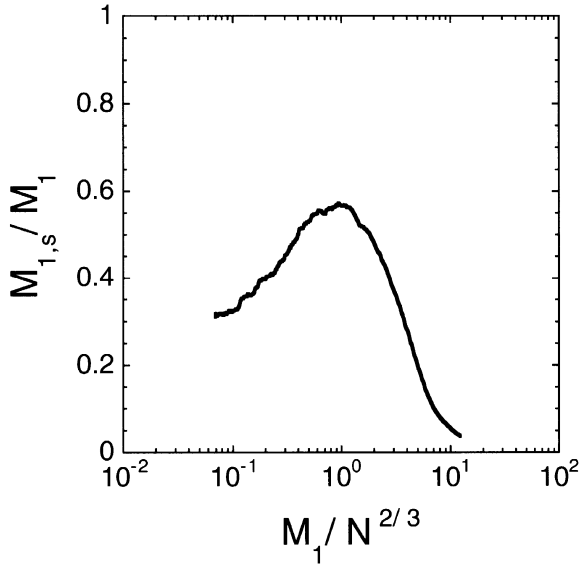

Fig. 8. The ratio of the standard deviation to the expectation value of the mass of the largest body as a function of $M_1/N^{2/3}$ (for the case of $A_{ij} = i \times j$). The other parameters are put to be $\delta = 1.2$ and $\epsilon = 0.01$.

would not be general but depends on the functional form of adopted A_{ij}. As to this point, we will discuss in Section 5.

5. The Case of Planetesimal Accumulation

In the previous section, we considered the case where the collisional probability A_{ij} is proportional to the square of mass, i.e., $A_{ij} = i \times j$ (note that we say, for example, A_{ij} is proportional to mass when $A_{ij} = i + j$). In this section, we derive a suitable set of δ and ϵ for describing the planetesimal accumulation process. As readily conjectured, when A_{ij} is proportional to a power of mass with a large exponent, runaway growth of one body starts early (Wetherill, 1990) and the mass of the runaway body increases rapidly. In such cases, the numerical simulation becomes hard and strong conditions would be needed for obtaining a precise numerical solution (speaking in our present language, δ_c and ϵ_c would become small). In order to confirm the validity of Wetherill and Stewart's method for the planetary accumulation process, we examine appropriate ranges of δ and ϵ for the collisional probability in the planetesimal accumulation.

If we neglect the effects of the solar gravity and assume a uniform distribution of planetesimals, we describe the planetesimal accumulation process in terms of the surface number density of planetesimals instead of the number density and then A_{ij} is approximately given by (Makino *et al.*, 1998)

$$A_{ij} = \frac{\pi}{\max(H_i, H_j)}(r_i + r_j)^2 \left\{1 + \frac{v_{\rm esc}^2}{v_{ij}^2}\right\} v_{ij}, \qquad (25)$$

where r_i is the radius of a planetesimal with mass i, v_{ij} is the relative velocity between bodies with masses i and j, and $v_{\rm esc}$ is the escape velocity given by (note that the unit mass is 1)

$$v_{\rm esc} = \sqrt{\frac{2G(i+j)}{r_i + r_j}}. \qquad (26)$$

The scale height of planetesimal swarm with mass i, H_i, is proportional to the mean random velocity v_i of bodies with mass i. We consider the low velocity case (i.e., $v_{ij} \ll v_{\rm esc}$) where gravitational focusing is effective. It is assumed that the mean random velocities of planetesimals are determined by the energy equipartition between them, i.e., $v_i \propto i^{-1/2}$ (Kokubo and Ida, 1996) and that the relative velocity v_{ij} is determined by the larger mean random velocity as $v_{ij} = \max(v_i, v_j)$. From the above assumptions, we have for A_{ij}

$$A_{ij} = \alpha\ \min(i, j)(i^{1/3} + j^{1/3})(i + j), \qquad (27)$$

where α is a certain constant independent of mass. The above collisional probability has the mass dependence as $A_{ij} \propto (mass)^{7/3}$ if bodies mainly captures the others with comparable masses to itself, which is satisfied for runaway bodies in this case.

As in the case of $A_{ij} = i \times j$, we scale the mass and time to describe the stage when runaway growth of one body starts. Makino *et al.* (1998) approximately solved the Smoluchowski equation for the collisional probability given by Eq. (27). Their approximate mass distribution is given by

$$\langle n_k \rangle = \frac{2}{3} N k^{-8/3} \qquad (28)$$

for masses k smaller than the high mass end of the distribution. The above mass distribution agrees well with the results of N-body simulation (Kokubo and Ida, 1996) and the statistical simulation (Wetherill and Stewart, 1989, 1993) for the case that collisional fragmentation is not considered. As the accumulation process proceeds, the high mass end of the mass distribution becomes large and the number of planetesimals in the high mass end decreases. As a result, the mass ratio of the largest planetesimal to the second largest one increases and runaway growth starts. We consider that runaway growth of one body starts when the largest body becomes twice massive as the second largest one. Using Eqs. (7) and (28), we roughly evaluate the mass of the largest body at the starting time as $M_1 \sim N^{3/5}$. The starting time is also evaluated as $\eta = Nt \simeq 0.2\alpha^{-1}$ from the Smoluchowski equation. Using Eq. (15), we obtain the growth rate of the largest body as

$$\frac{dM_1}{d\eta} = \frac{2^{6/5}}{5^{1/5}} \alpha^{-1} N^{1/5} M_1^{4/3}. \qquad (29)$$

As readily seen from the above estimation, it is convenient to introduce the following scaled mass $\tilde{M}$ and time u:

$$\tilde{M} = M/N^{3/5}, \qquad (30)$$

$$u = N^{2/5}(\alpha\eta - 0.2). \qquad (31)$$

If we use $\tilde{M}$ and u, the scaled growth rate of the largest body, $d\tilde{M}/du$, does not depend on the total mass N and α. That is, the growth curve of the largest and second largest bodies in the $u - M/N^{3/5}$ plane is almost independent of N.

In Fig. 9, we show the growth curves of the runaway bodies which are obtained from numerical simulations with various δ ($\delta = 1.2$, 1.5, and 2.0) and $\epsilon = 0.01$ (N being put to be

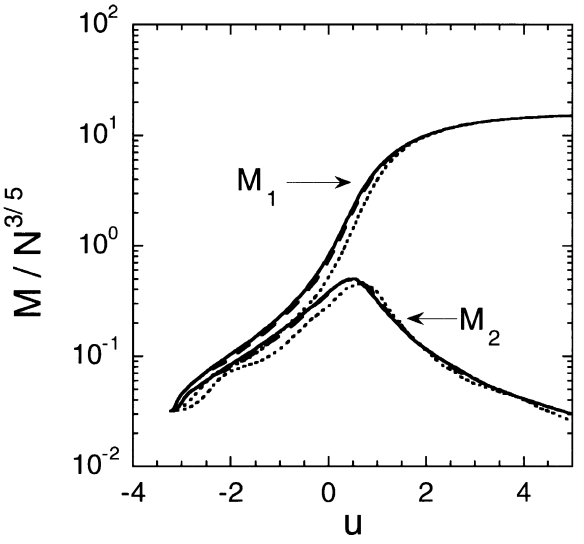

Fig. 9. The growth of the largest and the second largest bodies numerically found for the case of $A_{ij} = \alpha \times \min(i, j)(i^{1/3} + j^{1/3})(i + j)$. The vertical axis shows the mass normalized by $N^{3/5}$. The masses are obtained by ensemble means of 1000 runs and the abscissa is the renormalized time, u, given by Eq. (31). Solid, dashed, and dotted curves present the solutions obtained by the numerical simulations with $\delta = 1.2$, 1.5, and 2.0, respectively. Other parameters are put to be $\epsilon = 0.01$ and $N = 1 \times 10^3$.

1000). The simulations with $\delta = 1.2$ and 1.5 give almost the same growth curves of the largest and the second largest bodies. Hence we presume the precise solution to the stochastic coagulation equation. For the case of $\delta = 2$, the starting time of runaway growth of one body delays compared with that of the simulation with $\delta = 1.2$: the time at which the mass of the largest body becomes $N^{3/5}$ and the time at which the second largest body experiences its maximum mass are shifted toward positive u by a factor of 0.5. In the previous case (where $A_{ij} = i \times j$), we frequently observed the time delay due to large δ and, hence, such a feature would be general.

Now, we make a quantitative comparison by the same way as Section 4. Instead of X_a calculated from the analytical solution, we use the values X_c which is obtained from the simulation with $\delta = 1.1$ and $\epsilon = 0.01$. Since we know that the simulations with δ smaller than 1.5 give the almost same solutions and that the simulation with small δ can describe the accumulation process precisely, it would be proper to replace X_a by X_c. Demanding the level of accuracy given by inequality (21) and Eq. (22), we obtain the $\delta_c - \epsilon_c$ curve for the case of $N = 1000$ which is shown in Fig. 10. The $\delta_c - \epsilon_c$ curve behaves similarly with that of the case of $A_{ij} = i \times j$ (Fig. 5), though the values of δ_c and ϵ_c become small: especially, the value of ϵ_c becomes very small ($\epsilon_c = 0.07$) compared with the previous case even when δ is as small as 1.1 or 1.2 and, furthermore, decreases suddenly with an increase in δ when $\delta > 1.3$.

In Fig. 11, we illustrate how δ_c depends on the total mass of the body system N (ϵ being fixed to be 0.01) as well as for the case of ϵ_c (δ is fixed to be 1.2). As seen from the figures, δ_c and ϵ_c decrease with an increase in N and converge to 1.3 and 0.04, respectively, in the limit of large N. The behaviors of δ_c and ϵ_c are observed similarly to the case of $A_{ij} = i \times j$. However, compared with the limiting values of δ_c and ϵ_c for the case of $A_{ij} = i \times j$ ($\delta_c = 1.6$ and $\epsilon_c = 0.03$), δ_c is rather small in this case although ϵ_c is almost the same. This is due to the fact that A_{ij} depends strongly on the mass in this case.

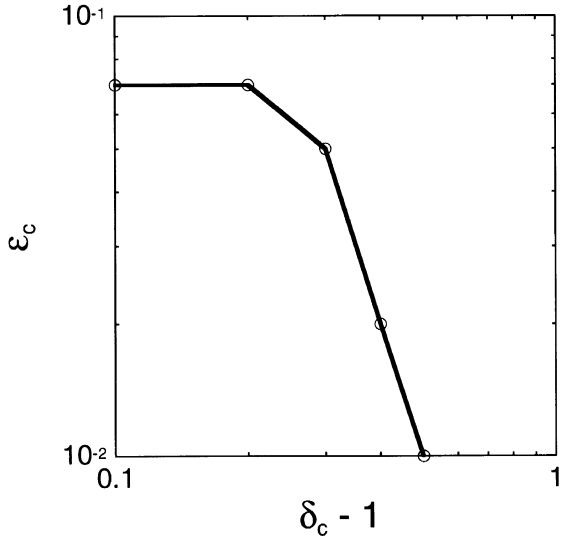

Fig. 10. The same as Fig. 5, but for the case of $A_{ij} = \alpha \times \min(i, j)(i^{1/3} + j^{1/3})(i + j)$.

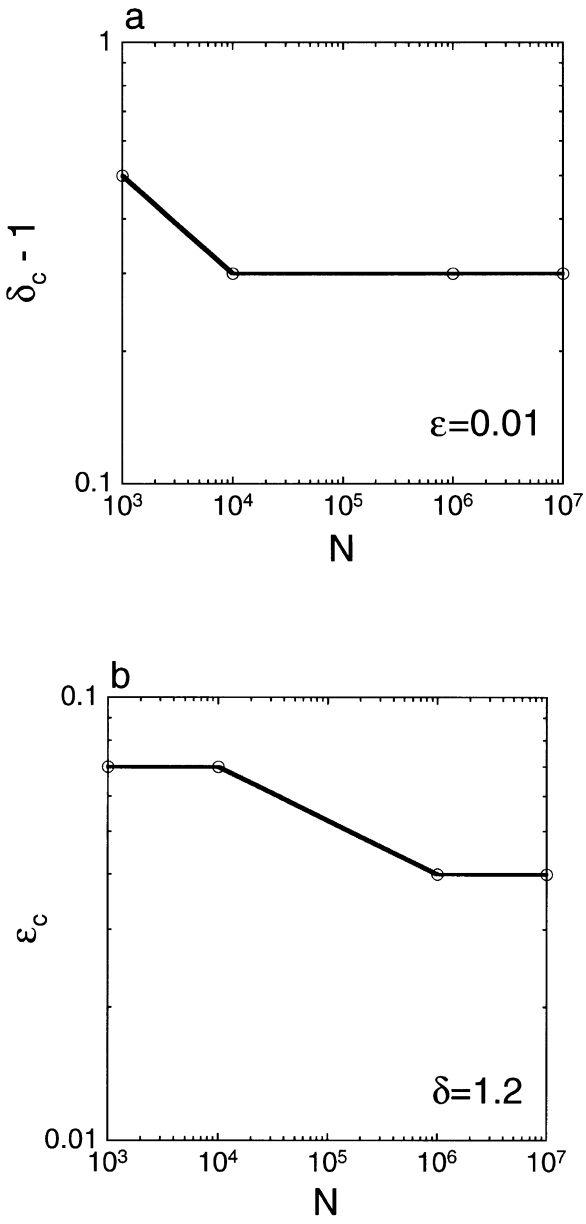

Fig. 11. The same as Fig. 6, but for the case of $A_{ij} = \alpha \times \min(i, j)(i^{1/3} + j^{1/3})(i + j)$.

Since we assume that the energy equipartition is attained between planetesimals, the collisional probability we adopted depends on mass as the power law with the largest exponent and, as a result, runaway growth becomes fast. On the other hand, Wetherill and Stewart (1989, 1993) calculate the time evolution of very broad mass distribution coupled with the velocity evolution. Their results show that such broad mass distributions do not evolve toward a simple power law and that the velocity is much weaker function of the mass than one would predict from energy equipartition. As a result, the exponent of the collisional probability in the real accumulation process becomes smaller than that of our simplified model. Therefore, in order to simulate, with sufficient accuracy, all the stages of the planetesimal accumulation, the adopted values of δ and ϵ must be smaller than 1.3 and 0.03, respectively.

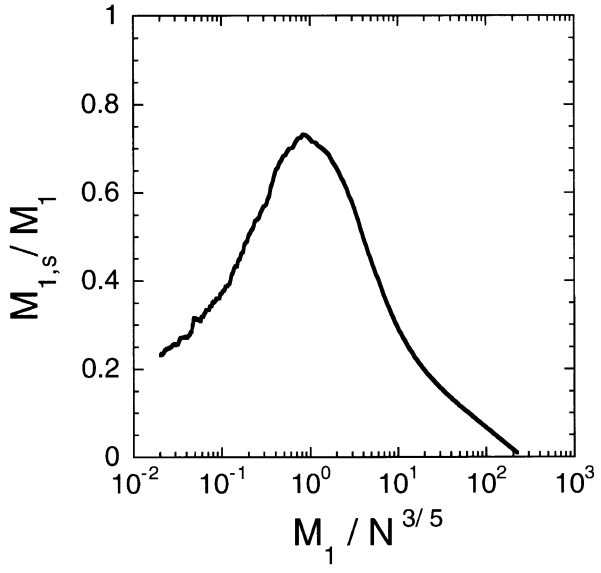

Fig. 12. The same as Fig. 8, but for the case of $A_{ij} = \alpha \times \min(i, j)(i^{1/3} + j^{1/3})(i + j)$.

The ratio of the standard deviation $M_{1,s}$ (given by Eq. (23)) to the mean value M_1 (given by Eq. (20)) of the mass of the largest body is illustrated in Fig. 12 as a function of $M_1/N^{3/5}$. In a qualitative sense this figure is quite similar to Fig. 8: the ratio has a maximum value in the vicinity of $M_1/N^{3/5} = 1$ which corresponds to $u = 0$ and decreases toward the direction of large $|u|$. However, the peak value of $M_{1,s}/M_1$ is as large as 0.75 in this case (in the case of $A_{ij} = i \times j$ the peak value is about 0.6). As mentioned in the Subsection 4.3, $M_{1,s}/M_1$ becomes large when A_{ij} is given by the power law of the mass with a high exponent. Thus, in the numerical simulation of the planetary accumulation by the use of WS89 method we must take the ensemble mean over about 500 independent simulations if we are interested, especially, in the planetary growth around the stage of beginning of runaway growth.

6. Conclusion and Discussion

In the present study we checked the validity and usefulness of the numerical method, proposed by Wetherill and Stewart (1989), for describing the late stage of planetary accumulation by comparing the numerical solution of WS89 method (or more exactly, the ensemble mean of a sufficiently large number of simulations) with the analytical solution (or the well-evaluated numerical solution). As mentioned in Section 3, our numerical code, which was reconstructed on the basis of WS89 method, is optimized by the two technical parameters δ and ϵ: δ is the ratio of the masses of adjacent mass batches and ϵ is the parameter determining the time interval in the numerical time integration.

We considered the two cases, $A_{ij} = i \times j$ and $\alpha \times \min(i, j)(i^{1/3} + j^{1/3})(i + j)$, as for the collisional probability. The latter case corresponds to the simplified model of the planetesimal accumulation. In the former case the analytic solutions to the stochastic coagulation equation is known while in the latter case no analytic solution is found yet. In both cases runaway growth starts at a certain time.

By the comparison between numerical and analytical solutions for the case of $A_{ij} = i \times j$, we confirmed that the ensemble mean obtained from WS89 method reproduces well the analytical solution when both δ and ϵ are sufficiently small. Furthermore, we found that the numerical simulation can describe the accumulation process within an admissible level of accuracy when the values of δ and ϵ are smaller than certain critical values (which are denoted by δ_c and ϵ_c, respectively). The critical values δ_c and ϵ_c are given by 1.6 and 0.03, respectively when the initial number of particles N, is large enough.

We examined the critical values δ_c and ϵ_c, in the simplified case of planetesimal accumulation (i.e., $A_{ij} = \alpha \times \min(i, j)(i^{1/3} + j^{1/3})(i + j)$). Since no analytical solution exist in this case, we adopted a numerical solution with sufficiently small δ and ϵ as a standard measure. Then, the critical values were obtained as $\delta_c = 1.3$ and $\epsilon_c = 0.04$, irrelevant of the initial number of particles. Compared with the case of $A_{ij} = i \times j$, δ_c is somewhat small while ϵ_c is almost the same. This would be due to strong dependence of A_{ij} on mass ($A_{ij} \propto (mass)^{7/3}$).

Since WS89 method contains the stochastic procedures, we examined how large the dispersion of the results is and how many runs are needed for obtaining a reliable ensemble mean. From the ratio of the standard deviation to the expectation value, which are obtained by the numerical simulations, we found that the demanded number of runs is as large as 400 (or 500) for the case of $A_{ij} = i \times j$ (or $A_{ij} = \alpha \times \min(i, j)(i^{1/3} + j^{1/3})(i + j)$) around the time when runaway growth starts. But, before that, the demanded number of runs is rather small (10 or so).

In real planetary accumulation, the dependence of the collisional probability A_{ij} on masses of colliding particles would be weaker than that we adopted ($A_{ij} \propto (mass)^{7/3}$). Generally, δ_c and ϵ_c become larger when A_{ij} depends weakly on the mass. Hence, we reach the following conclusions: WS89 method can describe the planetary accumulation process within a reasonable level of accuracy if we adopt the values of δ and ϵ smaller than 1.3 and 0.03, respectively, and if we evaluate the ensemble mean on the basis of 500 numerical simulations. Because Wetherill and Stewart (1989) and (1993) used the value of δ smaller than 1.1 and 1.2, respectively, they used their method correctly and, thus, give reliable results. On the other hand, since they did not perform sufficient averaging over an ensemble of similar accumulation process, divergence from the mean value for the growth of the largest body might be found. From the viewpoint of the numerical handling, the values of δ ($= 1.3$) is not so small comparing with those in other numerical simulations. In fact, when $\delta = 1.3$, we can cover the mass coordinate over 1×10^{20} mass range only with 180 mass batches. Thus, it is concluded that the new method proposed by Wetherill and Stewart is the valid and useful method for pursuing the planetary accumulation process.

In the present study, we showed that WS89 method can describe the time evolution of the mass distribution precisely. However, the real planetesimal accumulation process is more complex since the time evolution of the velocity distribution and the mass distribution should be calculated simultane-

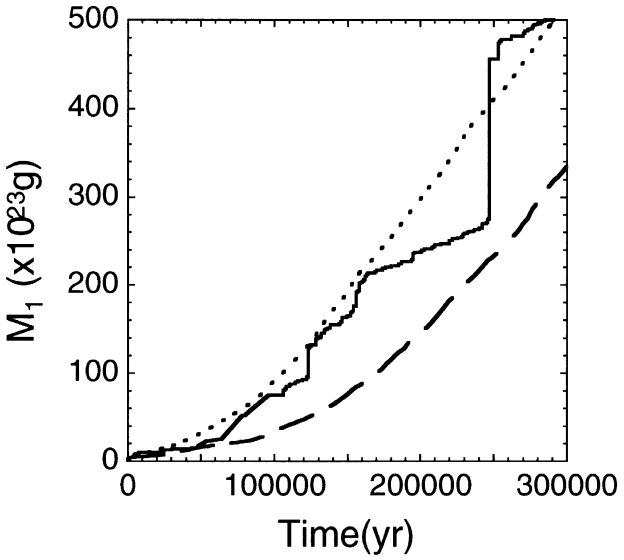

Fig. 13. The preliminary result of comparison between the statistical simulation and N-body simulation (solid curve). The dotted and dashed curves indicate the evolution of the largest body's mass calculated by WS89 method with $\delta = 1.1$ and 2.5, respectively.

ously. In the subsequent paper, we will complete our statistical code which simulate the accumulation process coupled with the velocity evolution and examine the accuracy of our statistical code, by comparing the results of the statistical code with those of N-body direct simulations. In Fig. 13, we show a preliminary comparison of our statistical code with N-body simulation by Kokubo and Ida (1998). As to the time evolution of the random velocity (i.e., eccentricities and inclinations), we adopted the formulation by Stewart and Ida (1998). The results of Greenzweig and Lissauer (1992) is used for the collisional probability between planetesimals. Initially the masses of all bodies are 1×10^{23} g and the total number of bodies is 3000. The dotted and dashed curves indicate the evolution of the largest body's mass calculated by the statistical code with $\delta = 1.1$ and 2.5, respectively, and the solid curve gives the result of N-body simulation. The mass of the largest body obtained by our statistical code changes smoothly since we take the ensemble mean and, while, N-body simulation gives the discontinuous growth because it is a single calculation. Except for the statistical fluctuations in the N-body simulation, our statistical code well reproduces the results of the N-body simulation in the case of $\delta = 1.1$. On the other hand, in our statistical simulation with $\delta = 2.5$, the growth of the largest body delays as shown in the previous cases. Therefore, even if we consider the accumulation process coupled with the velocity evolution, the criterion derived in this paper is available. The detail of our statistical code and the comparison with N-body simulations will be described in the subsequent paper.

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