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Procrustean solution of the 9-parameter transformation problem

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The Procrustean "matching bed" is employed here to provide direct solution to the 9-parameter transformation problem inherent in *geodesy*, *navigation*, *computer vision* and *medicine*. By computing the centre of mass coordinates of two given systems; scale, translation and rotation parameters are optimised using the Frobenius norm. To demonstrate the Procrustean approach, *three* simulated and one real geodetic network are tested. In the first case, a minimum three point network is simulated. The second and third cases consider the over-determined eight- and 1 million-point networks, respectively. The 1 million point simulated network mimics the case of an air-borne laser scanner, which does not require an isotropic scale since scale varies in the X, Y, Z directions. A real network is then finally considered by computing both the 7 and 9 transformation parameters, which transform the Australian Geodetic Datum (AGD 84) to Geocentric Datum Australia (GDA 94). The results indicate the effectiveness of the Procrustean method in solving the 9-parameter transformation problem; with case 1 giving the square root of the trace of the error matrix and the mean square root of the trace of the error matrix as 0.039 m and 0.013 m, respectively. Case 2 gives 1.13×10^{-12} m and 2.31×10^{-13} m, while case 3 gives 2.00×10^{-4} m and 1.20×10^{-5} m, which is acceptable from a laser scanning point of view since the acceptable error limit is below 1 m. For the real network, the values 6.789 m and 0.432 m were obtained for the 9-parameter transformation problem and 6.867 m and 0.438 m for the 7-parameter transformation problem, a marginal improvement by 1.14%.

Key words: Procrustes, 9-parameter transformation, least squares solution, Frobenius, singular value decomposition (SVD).

1. Introduction

The 9-parameter affine transformation is defined as the problem of determining three scale parameters in the X, Y, Z directions (diag$\{\mathbf{S}\} \in \mathbb{R}^{3\times3}$), three rotation parameters ($\mathbf{R} \in \mathbb{R}^{3\times3}$) and three translation parameters ($\mathbf{T} \in \mathbb{R}^{3\times1}$). It is an extension of the 7-parameter transformation problem (Bursa, 1962; Wolf, 1963), with the scales determined along the three axes X, Y, Z instead of the usual scalar value.

Suppose that the scale is not uniform within two or more conurations of points such that they vary within given dimensions. This is typical in the 9-parameter transformation problem. In this case, transforming the coordinates of one configuration to those of another will require, in addition to rotation and translation elements, the solution of scale parameters s_1, s_2, s_3 corresponding to the 3 (X, Y, Z) dimensions. The 9-parameter transformation is used in the fields of geodesy (Antonopoulos, 2003; Watson, 2006), navigation (Forsberg, 1991), medicine (Piperakis and Kumazawa, 2001; Pfefferbaum *et al.*, 2006; Sun *et al.*, 2007), computer image analysis (Ashburner and Friston, 1997) and surface modelling (Niederoest, 2003; Gruen and Akca, 2005). Furthermore, the 9-parameter transformation is included in several geodetic coordinate transformation packages, which use iterative approaches requiring initial starting values (e.g., Mathes, 2002; Fröhlich and Bröker, 2003). They can also find use in correcting for distortion where the rotation and translation elements have been determined exclusive of scales, e.g., in Featherstone and Vaníček (1999).

The solution of the transformation parameters has attracted a wide range of research, e.g., Späth (2004), who applies a numerical minimization technique, Papp and Szucs (2005), who apply a linearized least-squares method and Watson (2006), who applies the Gauss-Newton method. The Multidimensional Scaling (MDS) approach of Procrustes has also been widely used in recent studies.

Borg and Groenen (1997) define Multidimensional Scaling as a method that represents measurements of similarity (dissimilarity) of data as distances among points in a geometric space of low dimensionality. Let us consider a data set to consist of tests and that a correlation of tests is required. MDS can be used to represent these data in a plane such that their correlation can be studied. The closer together the points are (i.e., the shorter the distance between the points) the more correlated they are. MDS thus gives the advantage of graphical visualization of hidden adherent properties between objects. Procrustes is a procedure used in MDS to realise its goals. It is a tool of MDS concerned with fitting one configuration to another as close as possible. Put in matrix form, the Procrustes problem asks how closely a matrix $\mathbf{A}$ can be approximated by a second, given, matrix $\mathbf{B}$ which is multiplied by an orthogonal matrix $\mathbf{T}$. The advantage it enjoys over the conventional methods is that it does not rely on approximate starting values, and that it is not iterative in nature. It offers a direct solution to nonlinear transformation equations (see, e.g., Awange and Grafarend, 2005).

In the Procrustean method, the rotation matrix is first estimated using *singular value decomposition* (SVD) of the two sets of coordinates, and then the translation and scale factors are estimated using these estimated rotation parameters. Therefore, the rotation parameters are independent of the number of scale factors in the transformation.

Whereas the Procrustean approach has successfully been applied to solve the 7-parameter datum transformation (also known as the Helmert transformation) problem (e.g., Grafarend and Awange, 2000, 2003; Awange and Grafarend, 2005; Umeyama, 1991; Beinat and Crosilla, 2001, 2002), its application to solve the 9-parameter transformation problem has not been attempted. This has been due to the very nature of the Procrustean approach, which is designed to give isotropic dilation, three rotation and three translation parameters (Cox and Cox, 1994), whereas the 9-parameter transformation requires the solution of three scale parameters. In solving the 9-parameter transformation problem for example, Watson (2006) circumvents the limitation of the Procrustean solution by solving rotational elements using Gauss-Newton iteration.

Lingoes and Borg (1978) proposed the PINDIS (Procrustean Individual Differences Scaling) approach to scale individual similarities according to the dimensional salience model. The PINDIS approach obtains the estimates by minimizing the trace of the sum of squares of residuals, where such residuals are obtained by subtracting parameters in a given configurations from the centroid values. Due to failure of achieving an analytical solution, the PINDIS algorithm reverts to iterative procedures (see, e.g., Commandeur, 1991, p. 8). The shortcomings of PINDIS have been pointed out in Commandeur (1991) who goes a step further to provide a general solution for scaling factors. Commandeur (*ibid*, p. 34) achieves the general solution of scale factors through the use of a constraint such that the sum of squares about the origin of the scaled configurations remains equal to the sum of squares of the original configurations centered on the origin.

The common feature between the Lingoes and Borg (1978) and Commandeur (1991) approaches is that the scales are assumed to be uniform in a given configuration and solved throughout for the total number of configurations. Assuming $\mathbf{x}_j$ to be a column vector containing the coordinates of points $j = 1, \ldots, n$ in m-dimensional space, the scale is defined as s_j. For two configurations $j = 1, 2$ in $m = 3$ dimensions, these procedures will be solving for s_1 and s_2 scales, i.e., one scale parameter for each configuration. This is distinctly different from 9-parameter transformation studied here, which seeks to solve one scale parameter for each dimension, thus three in total, while the number of configurations is restricted to two.

This study attempts to provide a complete Procrustean solution of the 9-parameter transformation problem by solving the 3 scale parameters (s_1, s_2, s_3), 3 rotation and 3 translation parameters. The physical meaning of these scale parameters can be interpreted as follows: given three-dimensional coordinates in two systems, the rigid transformation from one system to another will involve rotating about and translating along the X-, Y-, and Z-axis. In some systems such as in photogrammetry and laser scanning, some discrepancies exists along these axes which can be overcome by applying different scales to each of the axes, hence the need for scale parameters. In addition, applying different scale parameters in geodetic coordinate transformations can better account for systematic observation errors or processes such as solid Earth deformation.

The advantages of using Procrustean transformation in the case of the 7-parameter transformation are its *direct approach*, which alleviates the need for linearization, initial starting values, and iterations inherent in least-squares solution (LSS), and also its *simplicity* compared to other approaches (e.g., Awange and Grafarend, 2005). Its success is mainly due to the fact that it is designed to give isotropic dilation, rotation and translation of rigid bodies. For a uniform scale therefore, as is the case in the 7-parameter transformation, the application is straightforward.

For the 9-parameter affine transformation, there is, however, the requirement to determine three scale parameters instead of one, thereby presenting a challenge to the Procrustean method designed for one scale factor. The present study presents the first attempt to apply Procrustes to solve the 9-parameter transformation problem. The approach reduces the coordinates of the two systems into their centre of mass, optimizes the scale in the sense of LSS with the rotation matrix determined by means of *singular value decomposition* (SVD) also known as *Eckert-Young decomposition* (Eckert and Young, 1936).

The study is organized as follows: In Section 2, the 9-parameter transformation problem is introduced and its Procrustean solution is presented in Section 3. Section 4 considers test examples, while Section 5 summarizes the results.

2. Definition of the Problem

Let us consider two coordinate configurations $\mathbf{A} \in \mathbb{R}^{n\times m}$ and $\mathbf{B} \in \mathbb{R}^{n\times m}$ consisting of n points in an m-dimensional Euclidean space. The dataset in $\mathbf{A}$ and $\mathbf{B}$ spaces are expressed by

$$\mathbf{A} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ & \cdots & \\ x_n & y_n & z_n \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ & \cdots & \\ X_n & Y_n & Z_n \end{bmatrix}. \tag{1}$$

The 7-parameter similarity or Helmert transformation problem is defined by

$$\mathbf{A} = \mathbf{B}\mathbf{R}s + \mathbf{v}\mathbf{T}^T, \tag{2}$$

where $\mathbf{R} \in \mathbb{R}^{3\times 3}$ is an orthonormal rotation matrix satisfying the orthogonality condition $\mathbf{R}^T\mathbf{R} = \mathbf{I}_3$. The rotation matrix is parameterized by either Cardan or Euler rotation angles (Awange and Grafarend, 2005, pp. 262–263) and the rotation convention of Soler (1998) is commonly used to define the direction of the rotations. $s \in \mathbb{R}$ is the scale factor also known as dilation, $\mathbf{T} \in \mathbb{R}^{3\times 1}$ is the translation vector and $\mathbf{v}$ is a vector of ones with length n. The 7-parameter transformation problem is now solved by determining the 7 transformation parameters, i.e., scale $s \in \mathbb{R}$, 3 rotation elements of $\mathbf{R} \in \mathbb{R}^{3\times3}$, and 3 translation parameters $\mathbf{T} \in \mathbb{R}^{3\times1}$. Various approaches for solving the 7-parameter transformation problem have been discussed, e.g., in Awange and Grafend (2005).

The 9-parameter transformation problem, on the other hand, concerns itself with the solution of

$$\mathbf{A} = \mathbf{BRS} + \mathbf{v}\mathbf{T}^T, \tag{3}$$

where the scalar parameter $s \in \mathbb{R}$ in Eq. (2) has been substituted with a diagonal matrix $\mathbf{S} \in \mathbb{R}^{3\times3}$ given by

$$\mathbf{S} = \begin{bmatrix} s_x & & \\ & s_y & \\ & & s_z \end{bmatrix}, \tag{4}$$

with s_x, s_y, s_z being the X, Y, Z scales in the directions, respectively. As opposed to the solution of Eq. (2), Eq. (3) entails the determination of nine parameters, i.e., three scales $\mathbf{S} \in \mathbb{R}^{3\times3}$, three rotations elements of $\mathbf{R} \in \mathbb{R}^{3\times3}$ and three translation parameters $\mathbf{T} \in \mathbb{R}^{3\times1}$.

3. Solution of the 9 Transformation Parameters

From Eqs. (1) and (3), let $\mathbf{b}_i = [X_i\ Y_i\ Z_i]$ and $\mathbf{a}_i = [x_i\ y_i\ z_i]$. In order to obtain optimum 9-transformation parameters, scale diag($\mathbf{S}$) $\in \mathbb{R}^{3\times3}$, rotation $\mathbf{R} \in \mathbb{R}^{3\times3}$ and translation $\mathbf{T} \in \mathbb{R}^{3\times1}$ are determined, such that the Frobenius norm is minimum, i.e., the sum of square of distances between points in $\mathbf{A} \in \mathbb{R}^{n\times m}$ and $\mathbf{B} \in \mathbb{R}^{n\times m}$ coordinate systems

$$D^2 = \sum_{i=1}^{n} (\mathbf{a}_i - \mathbf{b}_i\mathbf{RS} - \mathbf{T})^T (\mathbf{a}_i - \mathbf{b}_i\mathbf{RS} - \mathbf{T}). \tag{5}$$

Optimal translation vector estimation:

Considering $\mathbf{a}_0$ and $\mathbf{b}_0$ to be the center of mass (centroid) of the two coordinate configurations $\mathbf{A}$ and $\mathbf{B}$ derived from

$$\mathbf{a}_0 = \frac{1}{n}\sum_{i=1}^{n} \mathbf{a}_i, \qquad \mathbf{b}_0 = \frac{1}{n}\sum_{i=1}^{n} \mathbf{b}_i. \tag{6}$$

The error metric (Eq. (5)), in the centralised coordinate is given by

$$\begin{aligned} D^2 = \sum_{i=1}^{n} &[(\mathbf{a}_i - \mathbf{a}_0) - (\mathbf{b}_i - \mathbf{b}_0)\mathbf{RS} + \mathbf{a}_0 - \mathbf{b}_0\mathbf{RS} - \mathbf{T}]^T \\ &[(\mathbf{a}_i - \mathbf{a}_0) - (\mathbf{b}_i - \mathbf{b}_0)\mathbf{RS} + \mathbf{a}_0 - \mathbf{b}_0\mathbf{RS} - \mathbf{T}], \end{aligned} \tag{7}$$

which on expanding leads to (Cox and Cox, 1994)

$$\begin{aligned} D^2 = \sum_{i=1}^{n} &[(\mathbf{a}_i - \mathbf{a}_0) - (\mathbf{b}_i - \mathbf{b}_0)\mathbf{RS}]^T \\ &[(\mathbf{a}_i - \mathbf{a}_0) - (\mathbf{b}_i - \mathbf{b}_0)\mathbf{RS}] \\ &+ n(\mathbf{a}_0 - \mathbf{b}_0\mathbf{RS} - \mathbf{T})^T (\mathbf{a}_0 - \mathbf{b}_0\mathbf{RS} - \mathbf{T}). \end{aligned} \tag{8}$$

A close examination of Eq. (8) reveals that the last term is non-negative and contains the translational element $\mathbf{T} \in \mathbb{R}^{3\times1}$. D^2 can attain a minimum if the last term in Eq. (8) equals zero, i.e., if

$$\mathbf{T} = \mathbf{a}_0 - \mathbf{b}_0\mathbf{RS}. \tag{9}$$

Alternatively, from Eq. (3), $\mathbf{T} \in \mathbb{R}^{3\times1}$ can also be obtained by taking the mean of the translations, i.e.,

$$\mathbf{T}_i = \frac{1}{n}\sum_{i=1}^{n} (\mathbf{a}_i - \mathbf{b}_i\mathbf{RS}) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{a}_i - \frac{1}{n}\sum_{i=1}^{n} \mathbf{b}_i\mathbf{RS}, \tag{10}$$

which together with Eq. (6) leads to the translation elements in Eq. (9).

Optimal scale parameter estimation:

Assuming the centroids $\mathbf{a}_0 = \mathbf{b}_0 = \mathbf{0}$, then from Eq. (3),

$$\mathbf{A} = \mathbf{BRS}. \tag{11}$$

Let $\mathbf{f}$ be the error function incurred in rotating the system $\mathbf{B}$ and scaling it to match $\mathbf{A}$. Thus

$$\mathbf{f} = (\mathbf{A} - \mathbf{BRS})^T (\mathbf{A} - \mathbf{BRS}), \tag{12}$$

where $\mathbf{f}$ is a 3×3 matrix. As in the translation estimation, the Frobenius norm is applied to the error function by minimizing the distances between two corresponding points, and this norm is minimized (cf. Fitzpatrick and West, 2001)

$$\|\mathbf{f}\|_F^2 = \mathrm{tr}\{(\mathbf{A} - \mathbf{BRS})^T (\mathbf{A} - \mathbf{BRS})\} := \min \tag{13}$$

The optimal least squares solution of $\mathbf{S} \in \mathbb{R}^{3\times3}$ can be found by minimising the $\|\mathbf{f}\|_F^2$ such that

$$\left.\begin{aligned} &\frac{\partial\|\mathbf{f}\|_F^2}{\partial s_j} = 0, \ \text{necessary condition} \\ &\frac{\partial^2\|\mathbf{f}\|_F^2}{\partial s_j^2} \geq 0, \ \text{sufficient condition} \end{aligned}\right] \quad j \in \{1, 2, 3\}, \tag{14}$$

where s_1, s_2, and s_3 are the scale parameters in the direction of X, Y and Z, respectively. Since $\mathbf{S}$ is a diagonal matrix, and thereby symmetric, the following relations hold (Grafarend and Schaffrin, 1993, p. 443)

$$\begin{aligned} \frac{\partial\,\mathrm{tr}(\mathbf{S}\mathbf{R}^T\mathbf{B}^T\mathbf{A})}{\partial\mathbf{S}} &= \frac{\partial\,\mathrm{tr}(\mathbf{A}^T\mathbf{BRS})}{\partial\mathbf{S}} = \mathbf{R}^T\mathbf{B}^T\mathbf{A} \\ \frac{\partial\,\mathrm{tr}(\mathbf{S}\mathbf{R}^T\mathbf{B}^T\mathbf{BRS})}{\partial\mathbf{S}} &= 2\mathbf{R}^T\mathbf{B}^T\mathbf{BRS} \end{aligned} \tag{15}$$

Then, the first and second derivatives of $\|\mathbf{f}\|_F^2$ with respect to $\mathbf{S} \in \mathbb{R}^{3\times3}$ in Eq. (12) are

$$\left.\begin{aligned} \frac{\partial\|\mathbf{f}\|_F^2}{\partial\mathbf{S}} &= -2\mathbf{R}^T\mathbf{B}^T\mathbf{A} + 2\mathbf{R}^T\mathbf{B}^T\mathbf{BRS} \\ \frac{\partial^2\|\mathbf{f}\|_F^2}{\partial\mathbf{S}^2} &= 2\mathbf{R}^T\mathbf{B}^T\mathbf{BR} \end{aligned}\right] \tag{16}$$

Note that the sufficient condition in Eq. (14) is observed, since the second derivative in Eq. (16) is a quadratic form. By setting $\frac{\partial\|\mathbf{f}\|_F^2}{\partial\mathbf{S}} = 0$, the optimal least-squares solution for $\mathbf{S}$ can be found

$$\mathbf{S} = (\mathbf{R}^T\mathbf{B}^T\mathbf{BR})^{-1}\mathbf{R}^T\mathbf{B}^T\mathbf{A}. \tag{17}$$

However, the solution of Eq. (17) will generally result in a full **S** matrix instead of the desired diagonal matrix. To avoid this situation, such that the off-diagonal elements of **S** in Eq. (14) are zero, Eq. (16) needs to be evaluated term by term. Introducing the following term-wise notation of **A**, **B** and **R**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ & \cdots & \\ a_{n1} & a_{n2} & a_{n3} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ & \cdots & \\ b_{n1} & b_{n2} & b_{n3} \end{bmatrix},$$
$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{n1} & r_{n2} & r_{n3} \end{bmatrix}, \tag{18}$$

it is deduced (see Appendix) that the elements of $\frac{\partial \|\mathbf{f}\|_F^2}{\partial \mathbf{S}}$ are given by

$$\left\{\frac{\partial \|\mathbf{f}\|_F^2}{\partial \mathbf{S}}\right\}_{ij} = -2\sum_{l=1}^{n}\left(\sum_{k=1}^{3} b_{lk}r_{ki}\right)a_{lj} + 2s_j\sum_{l=1}^{n}\left(\sum_{k=1}^{3} b_{lk}r_{ki}\right)\left(\sum_{k=1}^{3} b_{lk}r_{kj}\right). \tag{19}$$

Setting the diagonal elements ($i = j$) to zero in Eq. (19) gives the solution for the scale parameters s_j

$$\left\{\frac{\partial \|\mathbf{f}\|_F^2}{\partial \mathbf{S}}\right\}_{jj} = 0 \Leftrightarrow s_j = \frac{\sum_{l=1}^{n}\left(\sum_{k=1}^{3} b_{lk}r_{kj}\right)a_{lj}}{\sum_{l=1}^{n}\left(\sum_{k=1}^{3} b_{lk}r_{kj}\right)^2}, \quad j \in \{1, 2, 3\}. \tag{20}$$

Note that this solution can also be found by expressing $\mathrm{tr}\{(\mathbf{A} - \mathbf{BRS})^T(\mathbf{A} - \mathbf{BRS})\}$ in term-wise notation, and then differentiating the result with respect to s_j.

In deriving Eq. (20), we made use of the assumption $\mathbf{a}_0 = \mathbf{b}_0 = \mathbf{0}$ which may not always hold due to poor network geometry. In order to take into consideration residual errors arising due to the assumption $\mathbf{a}_0 = \mathbf{b}_0 = \mathbf{0}$, the scale is rigorously solved as follows: Let the error in scale associated with the assumption $\mathbf{a}_0 = \mathbf{b}_0 = \mathbf{0}$ be $d\mathbf{S}$. A linear expression in **S** is then written as

$$\mathbf{S} = \mathbf{S}_0 + d\mathbf{S}, \tag{21}$$

where $\mathbf{S}_0$ is given by Eq. (20). This is then solved via least squares as

$$d\mathbf{S} = \mathrm{diag}((\mathbf{R}^T\mathbf{B}^T\mathbf{BR})^{-1}\mathbf{R}^T\mathbf{B}^T\mathbf{w}), \tag{22}$$

and

$$\mathbf{w} = \mathbf{A} - \mathbf{BRS}_0 - \mathbf{v}(\mathbf{a}_0 - \mathbf{b}_0\mathbf{RS}_0), \tag{23}$$

where the diagonal operator, diag, means that a matrix is 'diagonalized' by setting the non-diagonal elements to zero and $d\mathbf{S}$ is still a 3×3 matrix after this operation. Note that Eq. (20) is influenced by the assumption $\mathbf{a}_0 = \mathbf{b}_0 = \mathbf{0}$ and thus few iterations based on Eqs. (21)–(23) are generally required as shown in Section 4.

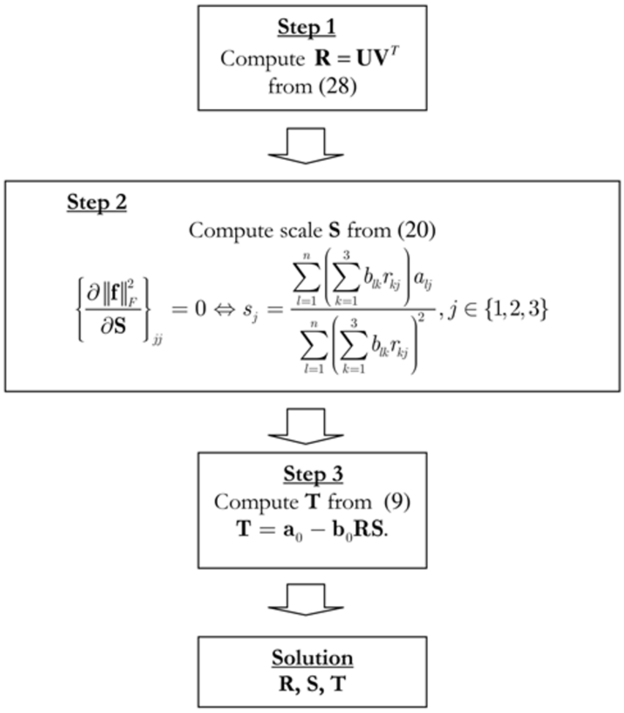


Fig. 1. Procrustean 9-parameter transformation algorithm.

Optimal rotation matrix estimation:

In Eqs. (9) and (17), the rotation matrix is required for solving the scale and translation elements. Here, we illustrate how it can be obtained from the partial orthogonal Procrustes method which makes use of only the coordinates in two systems and does not require linearization, iteration or the knowledge of approximate starting values. Consider the transformation problem about the origin $\mathbf{a}_0 = \mathbf{b}_0 = \mathbf{0}$. Since the solution of $\mathbf{R} \in \mathbb{R}^{3\times3}$ can be obtained from coordinates independent of the scale and translation parameters (Grafarend and Awange, 2003; Watson, 2006), Eq. (3) is expressed as

$$\mathbf{A} = \mathbf{BR}, \tag{24}$$

subject to the quadratic constraint and the orthogonality condition of the rotation matrix **R** (Grafarend and Awange, 2003)

$$\mathbf{R}^T\mathbf{R} = \mathbf{I}_3 \tag{25}$$

The 3×3 orthonormal matrix $\mathbf{R} \in \mathbb{R}^{3\times3}$ is a least-squares solution of Eq. (24) with the condition in Eq. (25), if and only if, it fulfils the system of nonlinear normal equation

$$\mathbf{B}^T\mathbf{BR} + \mathbf{R}\Lambda = \mathbf{B}^T\mathbf{A}, \tag{26}$$

where Λ is a symmetric 3×3 matrix of Lagrange multipliers (Grafarend and Awange, 2000).

If the singular value decomposition of $\mathbf{B}^T\mathbf{A}$ is expressed as

$$\mathbf{U}\Sigma\mathbf{V}^T = \mathbf{B}^T\mathbf{A}, \tag{27}$$

then a solution of the system of nonlinear normal equations is

$$\mathbf{R} = \mathbf{UV}^T \tag{28}$$

which is unique if the rank of $\mathbf{B}^T\mathbf{A} \in \mathbb{R}^{m\times m}$ is equal or greater than 3 (Grafarend and Awange, 2000). Other

Table 1. True and estimated transformation parameters for simulated 3 points network. Rota, Rotb and Rotc are the rotation angles around the x, y and z axes.

	True value	Estimated value	Difference
s_x	0.99998	0.99998	4.608314×10^{-11}
s_y	0.99994	0.99994	4.608192×10^{-11}
s_z	0.99995	0.99995	1.221245×10^{-15}
T_x (m)	100.0	100.019010	−0.019010
T_y (m)	20.0	19.975360	0.024640
T_z (m)	0.0	1.136868×10^{-13}	$-1.136868 \times 10^{-13}$
Rota (‘’)	1.0	0.999971	2.880136×10^{-5}
Rotb (‘’)	3.0	3.000010	-9.600299×10^{-6}
Rotc (‘’)	0.5	2.480232	−1.980232

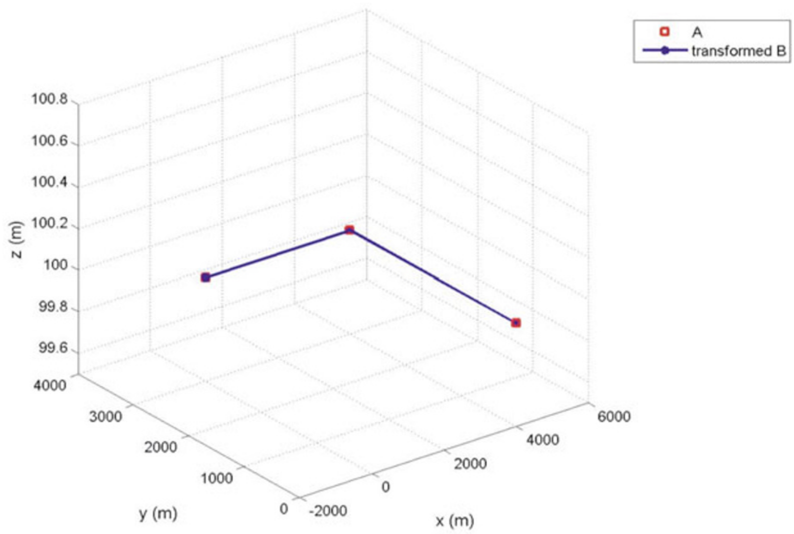


Fig. 2. Two simulated three dimensional 3-points networks.

methods of obtaining Eq. (28) have been presented, e.g., in Awange and Grafarend (2005). The Procrustean 9-parameter transformation algorithm is presented in Fig. 1.

4. Test Results

In this section, the proposed Procrustean 9-parameter transformation is tested using simulated datasets and a real network. Regarding the number of points in a test network, we utilize the minimum required number, i.e., 3, to a large number of points, 1 million as can be the case of the point cloud registration of air-borne laser scanner datasets.

The simulation adopts a “*forward-backward*” strategy, where we start with some arbitrary coordinates of **A** and values of the 9 transformation parameters. These are then used in the “*forward step*” to simulate the coordinates of configuration **B**. In the “*backward step*”, we use the simulated coordinates of **B** and those of **A** to derive the transformation parameters using the Procrustean 9-parameter algorithm. A comparison is then made between the ‘forward’ and ‘backward’ transformation parameters. The computed transformation parameters from the “*backward step*” are used to transform the coordinates of **B** into **A** and the errors are analysed using two error measures errE and MerrE, which are defined as (e.g., Grafarend and Awange, 2003)

$$\mathrm{errE} = \sqrt{\mathrm{tr}(\mathbf{E}^T\mathbf{E})} \tag{29}$$

and

$$\mathrm{MerrE} = \sqrt{\frac{\mathrm{tr}(\mathbf{E}^T\mathbf{E})}{3n}}, \tag{30}$$

where **E** is the error matrix

$$\mathbf{E} = \mathbf{A} - \mathbf{BRS} - \mathbf{v}(\mathbf{a}_0 - \mathbf{b}_0\mathbf{RS}). \tag{31}$$

The error matrix norm errE is thus the square root of the sum of the squared coordinate differences before and after transformation. This error measure is generally larger for cases with a larger number of points. The mean error matrix norm MerrE divides the sum of the squares by the total number of coordinates to obtain the root-mean-squares (RMS) of the coordinate differences, and is more suitable to compare cases with different numbers of points.

Case 1 (simulated 3 points network)

In this case, two networks consisting of 3 common points, which is the minimum required numbers of points for Procrustean 9-parameter transformation, are utilized as shown in Fig. 2. This example is important for a case where only the bare minimum number of points are available in both systems as is often the case in Photogrammetry. The transformation parameters between **A** and **B**, i.e., 3-scale factors, 3-translational and 3-rotational parameters are computed using the Procrustean 9-parameter algorithm and are presented in Table 1. The error measures errE and MerrE

Table 2. True and estimated transformation parameters for simulated 8 point network.

	True value	Estimated value	Difference
s_x	0.99998	0.99998	1.928983×10^{-8}
s_y	0.99994	0.99994	4.178106×10^{-9}
s_z	0.99995	0.99995	-6.129651×10^{-8}
T_x (m)	100.0	100.000000	$-1.421086 \times 10^{-14}$
T_y (m)	20.0	20.000000	8.526513×10^{-14}
T_z (m)	0.0	$-8.526513 \times 10^{-14}$	8.526513×10^{-14}
Rota ('')	1.0	0.993931	0.006069
Rotb ('')	3.0	3.013453	−0.013453
Rotc ('')	0.5	0.487267	0.012733

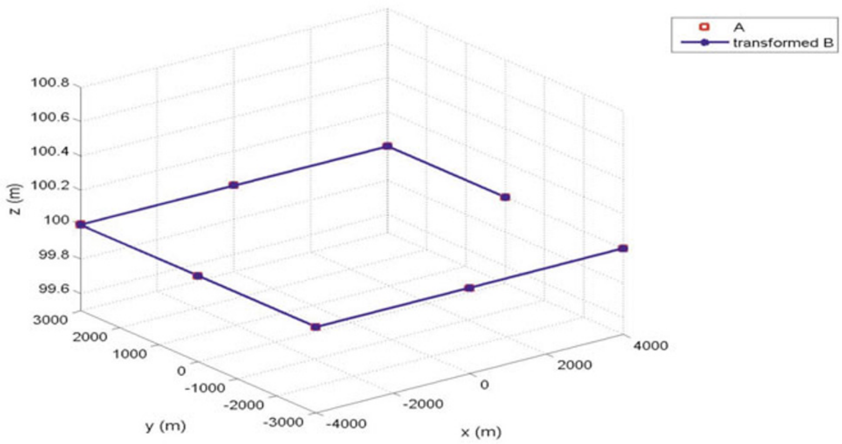


Fig. 3. Two simulated three dimensional 8-points networks.

computed using (Eq. (29)) and (Eq. (30)) are 0.0392 m and 0.0131 m, respectively. The difference between the true transformation parameters and those computed using Procrustean algorithm are presented in Table 1. The differences are close to zero for scale and rotation elements, except for the Z-axis rotation. The Z-axis rotation error is much larger than the other direction errors since the range of the Z-direction data is much smaller than that of either X- or Y-axis dataset in Case 1, which causes the difference in Rot(c) to be slightly larger. The differences found in T_x and T_y are indeed related to this. This is also attributed to the fact that we are dealing with the minimum case. As the number of stations increase, e.g., in Cases 2 and 3, the effect becomes much less. No iteration for scale was required for this case in order to obtain the final MerrE value of 0.013064 m since the network geometry is good.

Case 2 (8-points network)

Next, two networks with 8 points shown in Fig. 3 are utilized. This example depicts a localized surveying situation where a couple of points common in both systems are available. The error measures errE and MerrE are computed as 1.132×10^{-12} m and 2.311×10^{-13} m. The differences between the true and estimated values of the 9-transformation parameters are presented in Table 2 are close to zero. Compared to case 1, the differences in the simulated 8-points networks are much smaller due to a large number of redundant points and a better network configuration compared to the minimal case of 3 points highlighting the well known fact of utilizing as many points as possible. For this simulated network, no iteration was required similar to the simulated 3-point network to obtain a MerrE value of 2.311370×10^{-13} m.

Case 3 (1 million points network)

For this case, two networks of one million common points are simulated. This case can be regarded as an example of an air-borne laser scanner point clouds situation in which the proposed Procrustean algorithm for solving 9-parameter transformation parameters can be useful since it is likely to have anisotropic scale factors. The error measures errE and MerrE are computed as 3.83×10^{-4} m and 2.208×10^{-7} m. The true and estimated transformation parameters are given in Table 3. From the plot of errE and MerrE vs iteration presented in Fig. 4, four iterations are sufficient for this case.

Case 4 (82-common known stations in both AGD 84 and GDA 94)

Finally, a real network dataset based on 82 stations common in both AGD 84 and GDA 94 (Fig. 5) are used to compute the transformation parameters for Western Australia. The AGD is defined by the ellipsoid with a semi-major axis of 6378160 m and a flattening of 0.00335289 while the GDA is defined by an ellipsoid of semi-major axis 6378137 m and a flattening of 0.00335281 (Kinneen and Featherstone, 2004). Both 7- and 9-transformation parameters are computed using the general 7-parameter Procrustean algorithm and the proposed 9-parameter Procrustean algorithms respectively. The resultant error measures errE and MerrE are then compared. For the 7-

Table 3. True and estimated transformation parameters for the simulated network with 1 million points.

	True values	Estimated values	Difference
s_x	0.99998	0.99998	7.193998×10^{-10}
s_y	0.99994	0.99994	$-7.097811 \times 10^{-11}$
s_z	0.99995	0.99995	-1.064053×10^{-9}
T_x (m)	400.0	399.999999	5.075449×10^{-7}
T_y (m)	300.0	300.000000	-1.495559×10^{-7}
T_z (m)	5.0	5.000000	4.753149×10^{-7}
Rota ('')	1.0	1.000072	-7.164506×10^{-5}
Rotb ('')	3.0	3.000320	-3.19516×10^{-4}
Rotc ('')	0.5	0.500084	-8.387693×10^{-5}

	Error X (m)	Error Y (m)	Error Z (m)
Min.	-6.5690×10^{-7}	-5.9980×10^{-7}	-3.6550×10^{-7}
Max.	6.5630×10^{-7}	6.0090×10^{-7}	3.6430×10^{-7}
Average	-7.1977×10^{-12}	4.6118×10^{-12}	-1.1743×10^{-11}

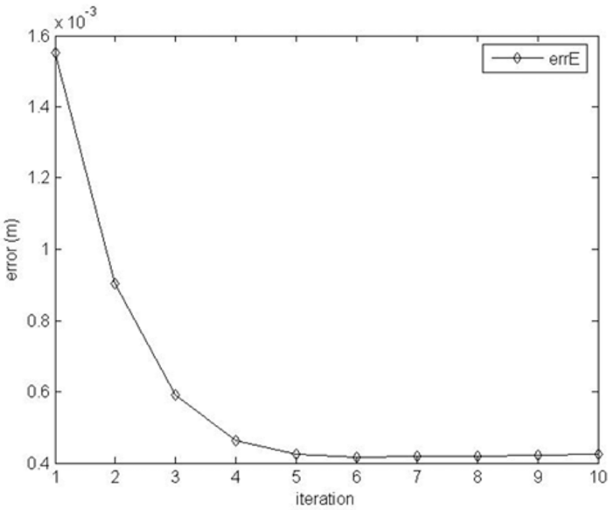


Fig. 4. Error versus iteration for the simulated network with 1 million points.

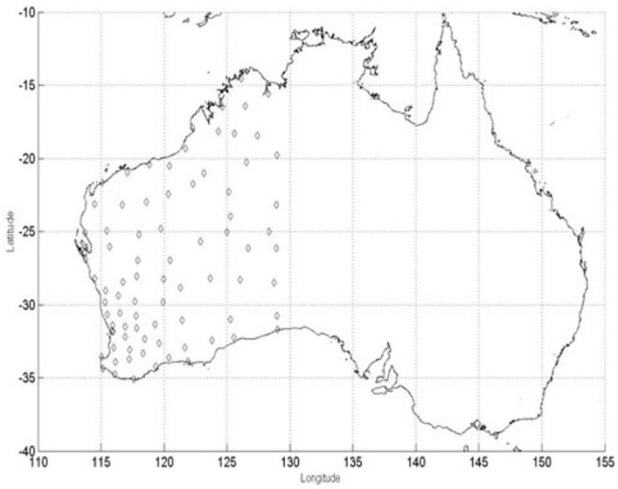


Fig. 5. Locations of the 82 stations in WA, Australia (AGD84 and GDA94).

Table 4. Error metrics from 9 and 7 parameters transformation with 82 stations in WA, Australia (AGD84 and GDA94).

	9-parameters	7-parameters	Improvement (%)
errE	6.788556	6.866908	1.14
MerrE	0.432822	0.437818	1.14

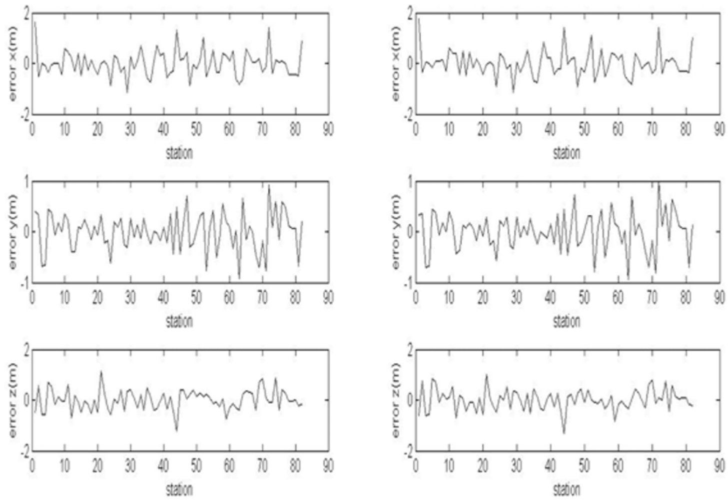
Table 5. Estimated transformation parameters between 82 stations in WA, Australia (AGD84 and GDA94) using 9- and 7-parameter transformation methods.

	Estimated values (9-parameters)	Estimated values (7-parameters)
s		1.00000368981
s_x	1.00000396085	
s_y	1.00000355062	
s_z	1.00000345982	
T_x (m)	−115.061896	−115.837707
T_y (m)	−47.697857	−48.373207
T_z (m)	144.095506	144.759551
Rota ('')	0.119711	0.119712
Rotb ('')	0.383988	0.383988
Rotc ('')	0.370396	0.370396

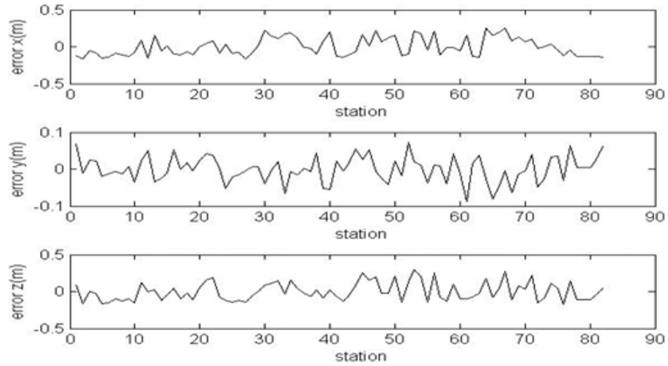
parameter solution, the error measures errE and MerrE are 6.867 m and 0.438 m, while the 9-parameter solution gives 6.789 m and 0.433 m which is a marginal 1.4% improvement on the results obtained with the 7-parameter transformation (see Table 4). The estimated transformation parameters are given in Table 5. The Error plots of each station for 82 stations in WA are presented in Fig. 6 while the plot of errE and MerrE versus iteration is presented in Fig. 7, which demonstrates that 4 iterations are required to obtain a precise scale matrix.

5. Conclusions

The test results of the Procrustean solution of the 9-parameter transformation demonstrate the effectiveness of the algorithm. In particular, its non requirement of approximate starting values or linearization inherent in the traditional least squares make it attractive. Even with many points, e.g., 1 million (case 3); all that is required of the user



(a) 9-parameters (b) 7-parameters



(c) Error difference between 9- and 7- parameters

Fig. 6. Error plots of each station for 82 stations in WA, Australia (AGD84 and GDA94) with 9- and 7-parameters.

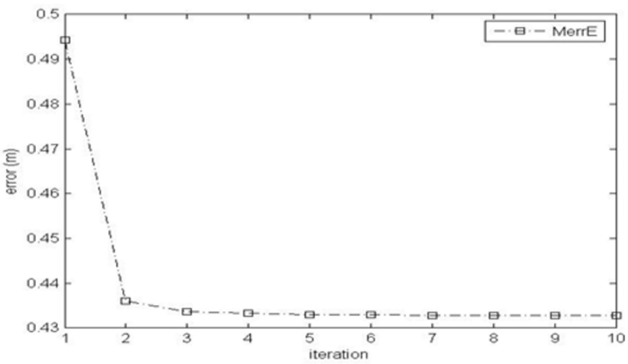


Fig. 7. Error versus iteration for 82 stations in WA, Australia (AGD84 and GDA94).

are the coordinates in both configurations to be set in matrix format. Compared to the general Procrustean algorithm used to solve the 7-parameter similarity transformation, this study has shown that a marginal improvement (i.e., 1.14% for the real network considered in this study) in the computed transformation parameters is gained when the scale parameters in the entire three axis are considered. This may be of use in larger geodetic network with many points, and where scale parameter cannot be assumed to be isotropic. The 9-parameter Procrustean algorithm considered in this study can thus be used for

(i) quicker and effective generation of 9 transformation parameters given coordinates in two systems as matrix configuration,
(ii) quick checking of the transformation parameters obtained from other methods
(iii) generating three scale parameters which could be useful in correcting distortions following procedures which first determine the rotation and translation parameters independent of scale e.g., (Featherstone and Vaníček, 1999).

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Appendix A. Proof of Eq. (19)

Given the matrices in Eq. (18)

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ & \cdots & \\ a_{n1} & a_{n2} & a_{n3} \end{bmatrix}, \qquad \mathbf{B}=\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ & \cdots & \\ b_{n1} & b_{n2} & b_{n3} \end{bmatrix},$$
$$\mathbf{R}=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{n1} & r_{n2} & r_{n3} \end{bmatrix}, \tag{A.1}$$

it can easily be found that the elements of the matrix product of **B** and **R** are given by

$$\{\mathbf{BR}\}_{ij}=\sum_{k=1}^{3} b_{ij} r_{kj}. \tag{A.2}$$

The elements of the transpose of **BR** simply follow by interchanging the indices i and j. Subsequently multiplying the transpose of **BR** by **A** gives

$$\{\mathbf{R}^T\mathbf{B}^T\mathbf{A}\}_{ij}=\sum_{l=1}^{n}\left(\sum_{k=1}^{3} b_{lk} r_{ki}\right) a_{lj} \tag{A.3}$$

and pre-multiplying **BR** by its transpose gives

$$\{\mathbf{R}^T\mathbf{B}^T\mathbf{BR}\}_{ij}=\sum_{l=1}^{n}\left(\sum_{k=1}^{3} b_{lk} r_{ki}\right)\left(\sum_{k=1}^{3} b_{lk} r_{kj}\right). \tag{A.4}$$

Multiplication of $\mathbf{R}^T\mathbf{B}^T\mathbf{BR}$ by a diagonal matrix $\mathbf{S}=\mathrm{diag}(s_1, s_2, s_3)$ yields

$$\{\mathbf{R}^T\mathbf{B}^T\mathbf{BRS}\}_{ij}=s_j\sum_{l=1}^{n}\left(\sum_{k=1}^{3} b_{lk} r_{ki}\right)\left(\sum_{k=1}^{3} b_{lk} r_{kj}\right). \tag{A.5}$$

Finally, inserting Eqs. (A.3) and (A.5) into Eq. (16) gives Eq. (19)

$$\begin{aligned}\left\{\frac{\partial\|\mathbf{f}\|_F^2}{\partial\mathbf{S}}\right\}_{ij} &= -2\{\mathbf{R}^T\mathbf{B}^T\mathbf{A}\}_{ij}+2\{\mathbf{R}^T\mathbf{B}^T\mathbf{BRS}\}_{ij} \\ &= -2\sum_{l=1}^{n}\left(\sum_{k=1}^{3} b_{lk} r_{ki}\right) a_{lj} \\ &\quad + 2s_j\sum_{l=1}^{n}\left(\sum_{k=1}^{3} b_{lk} r_{ki}\right)\left(\sum_{k=1}^{3} b_{lk} r_{kj}\right).\end{aligned} \tag{A.6}$$

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