Earth, Planets and Space

FULL PAPER

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A method for mixed additive and multiplicative random error models with inequality constraints in geodesy

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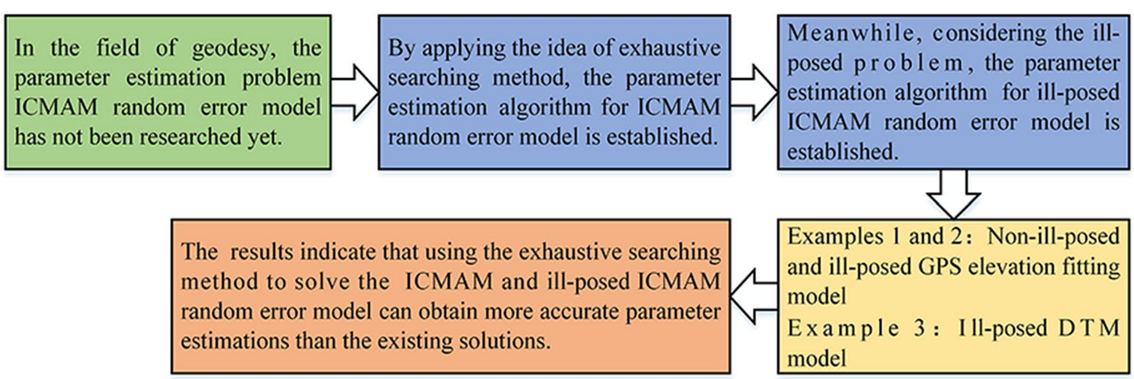
Abstract

In the geodetic data processing field, most methods for dealing with inequality constraints model are based on additive random error (ARE) models, and there have been few studies on mixed additive and multiplicative random error (MAAMRE) models with inequality constraints. To address this problem, a MAAMRE model with inequality constraints is first established based on the definition of inequality constraint equations, and then, a corresponding parameter estimation algorithm is proposed based on the idea of an exhaustive search method. In addition, considering a MAAMRE model for an ill-posed problem, an iterative regularization solution for an ill-posed MAAMRE model is first derived, and then, a specific parameter estimation algorithm for an ill-posed MAAMRE model with inequality constraints is further proposed by applying the exhaustive search approach. Finally, the feasibility and advantages of the proposed algorithms are verified by global positioning system (GPS) elevation fitting model and digital terrain model (DTM) examples.

Keywords: Mixed additive and multiplicative random error model, Inequality constraints, Equality constraints, Exhaustive search method, Ill-posed problem

Graphical Abstract



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Introduction

Random error models are the key to research on the characteristics of geodetic data and surveying adjustment (Yang and Zhang 2009), and traditional surveying adjustment theories and methods have been developed by treating the random errors of observations as additive random errors (AREs). However, with the rapid development of modern surveying and mapping technology, the random errors of the observations with electromagnetic waves as the carriers have begun to manifest as multiplicative random errors (MREs) or mixed additive and multiplicative random errors (MAAMREs) (Shi et al. 2014, 2015). For example, the speckle noise of synthetic aperture radar (SAR) manifests as multiplicative disturbance; the random errors of light detection and ranging (LiDAR) observations are proportional to the size of the observations; the baseline errors of global navigation satellite system (GNSS) and very long baseline interferometry (VLBI) are proportional to the length of the baselines; and the multiplicative and additive constants of electronic distance measurement (EDM) are essentially MAAMRE-type disturbances (Xu 1999; Shi 2012, 2014; Xu et al. 2013; Shi et al. 2014, 2015; He et al. 2019; Shi and Xu 2021; Wang et al. 2021; Wang and Chen 2021a, b; Wang and Han 2022). It can be seen that the random errors of these observations are related to their own true values (Shi et al. 2015); however, traditional surveying adjustment theories and methods always assume that they are independent. Therefore, research on corresponding theories and methods for MAAMRE models is an unavoidable and important scientific research task and has become an issue of great interest in the surveying and mapping field (Shi et al. 2014, 2015).

MAAMRE models are developed on the basis of MRE models. Although MRE models have been the focus of many theoretical studies and applications in the fields of statistics and geodesy (Shi et al. 2014, 2015), few studies have been conducted on MAAMRE models (Xu et al. 2013; Shi 2014; Shi and Xu 2021; Wang and Chen 2021a, b; Wang and Han 2022). Among them, Xu et al. (2013) first defined the functions and stochastic models for a MAAMRE model, derived the least squares solution and weighted least squares iterative solution based on the least squares principle, and extended the bias-corrected weighted least squares iterative solution for an MRE model given by Xu and Shimada (2000) to the MAAMRE model. Shi (2014) applied the three solutions in Xu et al. (2013) to LiDAR-type digital terrain model (DTM) data processing. Shi and Xu (2021) combined MRE and MAAMRE models to propose a more general MRE model with trends. Wang and Han (2022) derived a simple iterative method for a MAAMRE model with inequality constraints. Focusing on ill-posed MAAMRE models, Wang and Chen (2021a, b) researched the parameter estimation problems for an ill-posed MAAMRE model without constraints and an ill-posed MAAMRE model with equality constraints. However, few studies have been conducted on the parameter estimation problem for a MAAMRE model with inequality constraints. In recent years, as the understanding of the physical and mechanical properties of observed objects has become increasingly adequate, it has often become possible to obtain reasonable prior inequality constraint information between parameters before adjustment, and incorporating this prior constraint information into the function model during the adjustment process can significantly improve the accuracy and reliability of parameter estimation (Xie 2014). Therefore, there is an urgent need for research on the parameter estimation method for a MAAMRE model with inequality constraints.

At present, Gauss–Markov (GM), errors-in-variables (EIV) and partial error-in-variables (PEIV) models with inequality constraints have been the subjects of many studies. For example, Chiew (1976) and Peng et al. (2006) researched the parameter estimation problem for a GM model with inequality constraints, Zhang et al. (2013) researched the parameter estimation problem for an EIV model with inequality constraints, and Zeng et al. (2015) researched the parameter estimation problem for a PEIV model with inequality constraints. Among them, Zhang et al. (2013) first applied the idea of an exhaustive search method to propose an algorithm to solve the parameter estimation problem for the EIV model with inequality constraints, in which all possible parameter solutions are calculated by transforming the inequality constraint equations into equality constraint equations based on the definitions of active and inactive constraints and the optimal parameter solution is then selected. Subsequently, Xie (2014) further applied the idea of an exhaustive search method to the parameter estimation problem for a GM model with inequality constraints and proved that the exhaustive search method can yield an exact solution. As seen, the existing research on exhaustive search method has mainly focused on solving the parameter estimation problems for GM and EIV models with inequality constraints, and how to use this approach to solve the parameter estimation problem for a MAAMRE model with inequality constraints remains to be studied.

Starting from obtaining more accurate parameter estimates and aiming at the existing solutions for MAAMRE models that do not consider the prior inequality constraint information between the parameters during adjustment, this paper derives a parameter estimation algorithm for a MAAMRE model with inequality constraints. Meanwhile, considering a MAAMRE model for an ill-posed problem, the flowchart of a parameter estimation algorithm for an ill-posed MAAMRE model with inequality constraints is also given. Finally, the feasibility and advantages of the proposed algorithms are verified by examples.

Mixed additive and multiplicative random error model

Different from a GM model (Shi et al. 2014, 2015), which assumes AREs, a MAAMRE model can be expressed as follows (Xu et al. 2013):

$$\boldsymbol{y} = (\boldsymbol{A\beta}) \odot (1 + \boldsymbol{\varepsilon}_m) + \boldsymbol{\varepsilon}_a \tag{1}$$

where $\boldsymbol{y} \in \boldsymbol{R}^{n\times1}$ represents the observation vector disturbed by MAAMREs, $\boldsymbol{A} \in \boldsymbol{R}^{n\times t}$ represents a coefficient matrix, $\boldsymbol{\beta} \in \boldsymbol{R}^{t\times1}$ represents an unknown parameter vector, $\odot$ represents the Hadamard product of two matrices or vectors, $\mathbf{1} \in \boldsymbol{R}^{n\times1}$ represents a unit column vector, $\boldsymbol{\varepsilon}_m \in \boldsymbol{R}^{n\times1}$ represents a MRE vector that obeys a normal distribution, and $\boldsymbol{\varepsilon}_a \in \boldsymbol{R}^{n\times1}$ represents an ARE vector that obeys a normal distribution.

As noted by Shi (2014), for any variance matrices $\boldsymbol{D}_m = \sigma_m^2\boldsymbol{Q}_m$ and $\boldsymbol{D}_a = \sigma_a^2\boldsymbol{Q}_a$ of multiplicative and additive random errors, we can always use Cholesky decomposition to convert $\boldsymbol{Q}_m$ and $\boldsymbol{Q}_a$ into unit matrices when $\boldsymbol{Q}_m$ and $\boldsymbol{Q}_a$ are both positive definite matrices. Therefore, without loss of generality, this paper assumes that the multiplicative and additive random error vectors are independent of each other and that each has the same variance (Xu et al. 2013; Shi 2014; Shi et al. 2015; Shi and Xu 2021; Wang and Chen 2021a, b), namely, $\boldsymbol{D}_m = \sigma_m^2\boldsymbol{I}_n$, $\boldsymbol{D}_a = \sigma_a^2\boldsymbol{I}_n$ and $\mathrm{cov}(\boldsymbol{\varepsilon}_m, \boldsymbol{\varepsilon}_a) = 0$. Then, according to the definitions of the expectation and covariance matrix (Surveying Adjustment Group of School of Geodesy and Geomatics 2014), the expectation and covariance matrix of observation $\boldsymbol{y}$ can be obtained from Eq. (1) as follows:

$$\boldsymbol{E}(\boldsymbol{y}) = \boldsymbol{A\beta} \tag{2}$$

$$\boldsymbol{D}(\boldsymbol{y}) = \boldsymbol{E}[(\boldsymbol{y} - \boldsymbol{E}(\boldsymbol{y}))(\boldsymbol{y} - \boldsymbol{E}(\boldsymbol{y}))^{\mathrm{T}}] = \boldsymbol{M}_{ma}\boldsymbol{D}_m\boldsymbol{M}_{ma}^{\mathrm{T}} + \boldsymbol{D}_a \tag{3}$$

where $\boldsymbol{M}_{ma} = diag(\boldsymbol{A\beta})$.

Furthermore, the weight of observation $\boldsymbol{y}$ can be obtained as follows:

$$\boldsymbol{P}(\boldsymbol{y}) = \boldsymbol{Q}(\boldsymbol{y})^{-1} = \left(\frac{\boldsymbol{D}(\boldsymbol{y})}{\sigma_0^2}\right)^{-1} \tag{4}$$

where σ_0^2 represents the unit weight variance and $\boldsymbol{Q}(\boldsymbol{y})$ represents the cofactor matrix of the observation.

To facilitate subsequent formula derivations, we convert Eq. (1) as follows:

$$\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{A\beta} = (\boldsymbol{A\beta}) \odot \boldsymbol{\varepsilon}_m + \boldsymbol{\varepsilon}_a \tag{5}$$

where $\boldsymbol{e} \in \boldsymbol{R}^{n\times1}$ represents the residual vector of observation $\boldsymbol{y}$.

According to the cofactor propagation law, the cofactor matrix of $\boldsymbol{e}$ can be obtained from Eq. (5) as follows:

$$\boldsymbol{Q}(\boldsymbol{e}) = \boldsymbol{Q}(\boldsymbol{y}) = \boldsymbol{P}(\boldsymbol{y})^{-1} \tag{6}$$

As can be seen, the cofactor matrices $\boldsymbol{Q}(\boldsymbol{y})$ and $\boldsymbol{Q}(\boldsymbol{e})$ are both the nonlinear functions of the parameter estimates, which is obviously different from the case of a traditional ARE model. Thus, further research on MAAMRE models is needed.

Mixed additive and multiplicative random error model with inequality constraints and its solution

At present, models with inequality constraints are widely used in many contexts. For example, by adding appropriate inequality constraint information to the location and ambiguity parameters, the success rate of an ambiguity search can be significantly improved (Li and Shen 2009), and by adding appropriate inequality constraint information to the control point in dam deformation monitoring, more realistic deformation results can be obtained (Song 2019). Therefore, it is necessary to research the parameter estimation problem for a MAAMRE model with inequality constraints. On the basis of the observation equations of the GM model with inequality constraints defined in Xie (2014) and the EIV model with inequality constraints defined in Zhang et al. (2013), this paper establishes the following observation equations for a MAAMRE model with inequality constraints:

$$\begin{cases} \boldsymbol{e} = \boldsymbol{y} - \boldsymbol{A\beta} \\ \boldsymbol{G\beta} \leq \boldsymbol{W} \end{cases} \tag{7}$$

where $\boldsymbol{G} \in \boldsymbol{R}^{l\times t}$ represents the inequality constraint matrix, which is a full-rank matrix, and $\boldsymbol{W} \in \boldsymbol{R}^{l\times1}$ represents the constant vector of the inequality constraint matrix.

As can be seen, the parameter solution for the MAAMRE model with inequality constraints must satisfy the following two conditions:

$$\begin{cases} \boldsymbol{e}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{e})^{-1}\boldsymbol{e} = \min \\ \boldsymbol{G\beta} \leq \boldsymbol{W} \end{cases} \tag{8}$$

Furthermore, substituting the optimal parameter solution $\hat{\boldsymbol{\beta}}$ of Eq. (8) into the inequality constraint equation $\boldsymbol{G\beta} \leq \boldsymbol{W}$ yields the following two cases:

$$\begin{cases} \boldsymbol{G}\hat{\boldsymbol{\beta}} = \boldsymbol{W} \\ \boldsymbol{G}\hat{\boldsymbol{\beta}} < \boldsymbol{W} \end{cases} \tag{9}$$

For the first case of $\boldsymbol{G}\hat{\boldsymbol{\beta}} = \boldsymbol{W}$, the optimal parameter solution $\hat{\boldsymbol{\beta}}$ falls just on the boundary of the inequality constraint; such a case is an active constraint, which is also called an effective constraint. For the second case of $\boldsymbol{G}\hat{\boldsymbol{\beta}} < \boldsymbol{W}$, the optimal parameter solution $\hat{\boldsymbol{\beta}}$ falls within the boundary of the inequality constraint; such a case is an inactive constraint, which is also called an ineffective constraint. When no effective constraint exists, a MAAMRE model with inequality constraints is equivalent to the corresponding unconstrained MAAMRE model. At this time, the optimal parameter solution $\hat{\boldsymbol{\beta}}$ of the MAAMRE model with inequality constraints is equal to the parameter solution of the unconstrained MAAMRE model. When effective constraints do exist, a MAAMRE model with inequality constraints is equivalent to the corresponding MAAMRE model with equality constraints. At this time, the optimal parameter solution $\hat{\boldsymbol{\beta}}$ of the MAAMRE model with inequality constraints is equal to the parameter solution of the MAAMRE model with equality constraints, and we can establish the following observation equations for the MAAMRE model with equality constraints:

$$\begin{cases} \boldsymbol{e} = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{\beta} \\ \boldsymbol{G}_1\boldsymbol{\beta} = \boldsymbol{W}_1 \end{cases} \tag{10}$$

where $\boldsymbol{G}_1 \in \boldsymbol{R}^{q\times t}(0 \leq q \leq l)$ represents the effective constraint matrix and $\boldsymbol{W}_1 \in \boldsymbol{R}^{q\times 1}$ represents the constant vector of the effective constraint matrix.

According to the principle of the Lagrangian extremum value, Wang and Chen (2021b) derived a weighted least squares iterative solution for a MAAMRE model with equality constraints; the specific steps of iterative can be seen in Wang and Chen (2021b). Similarly, according to the definitions of active and inactive constraints (Lu et al. 1993; Feng et al. 2007), we can establish a parameter estimation algorithm for a MAAMRE model with inequality constraints based on the idea of an exhaustive searching method (Zhang et al. 2013; Xie 2014), as summarized in Algorithm 1. The specific steps are as follows:

1) The number of effective constraint sets is initialized as $k(k=0)$.
2) According to the inequality constraint equation in Eq. (7), we can obtain the total number of combinations of inequality constraint equations:

$$N = C_l^1 + C_l^2 + \cdots + C_l^l = \sum_{i=1}^{l} C_l^i \tag{11}$$

3) We select one combination sequentially from the N combinations of inequality constraint equations and assume that the selected equations are effective constraints. Then, by converting the selected inequality constraint equations into the corresponding equality constraint equations, we obtain the corresponding parameter estimates through the weighted least squares iterative solution for a MAAMRE model with equality constraints proposed by Wang and Chen (2021b).
4) We let $i = i + 1$ and substitute the parameter estimates into the selected inequality constraint equations. When the parameter estimates satisfy these equations, they are judged as an effective set of constraints, and we let $k = k + 1$; otherwise, they are judged as ineffective constraints, and the value k remains unchanged.
5) Steps 2)–4) are repeated until $i > N$ is satisfied, at which time the number of effective constraint sets k is output.

According to the above steps, the flowchart of Algorithm 1 can be obtained, which is shown in Fig. 1.

According to the above analysis, we can obtain the following three conclusions regarding the value of k:

1) When $k = 0$ is satisfied, all sets of inequality constraint equations are ineffective constraints. At this time, the optimal parameter solution $\hat{\boldsymbol{\beta}}$ for the MAAMRE model with inequality constraints is equal to the parameter solution for the corresponding unconstrained MAAMRE model, which is also the bias-corrected weighted least squares iterative solution proposed by Xu et al. (2013).
2) When $k = 1$ is satisfied, there exists a single of effective constraint equations. At this time, the optimal parameter solution $\hat{\boldsymbol{\beta}}$ for the MAAMRE model with inequality constraints is equal to the parameter solution obtained based on this set of effective constraint equations.
3) When $k > 1$ is satisfied, multiple sets of effective constraint equations exists. At this time, there are kk sets of parameter solutions, and we need to select the optimal parameter solution $\hat{\boldsymbol{\beta}}$ for the MAAMRE model with inequality constraints. Considering that the k sets of parameter solutions are all obtained via the weighted least squares iterative solution, in this paper, the k sets of parameter solutions are substituted into the weighted square sum of the residuals $\phi = \boldsymbol{e}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{e})^{-1}\boldsymbol{e}$ in turn, and the parameter solution $\hat{\boldsymbol{\beta}}$ corresponding to the minimum value $\phi_{\min}$ is selected as the optimal parameter solution for the MAAMRE model with inequality constraints based on the method in Xie (2014).

As can be seen, the principle of the exhaustive searching method is simple, applicable, and easy to understand and code, and this method can theoretically yield the optimal parameter solution for a model with inequality constraints and can be effectively used to solve the

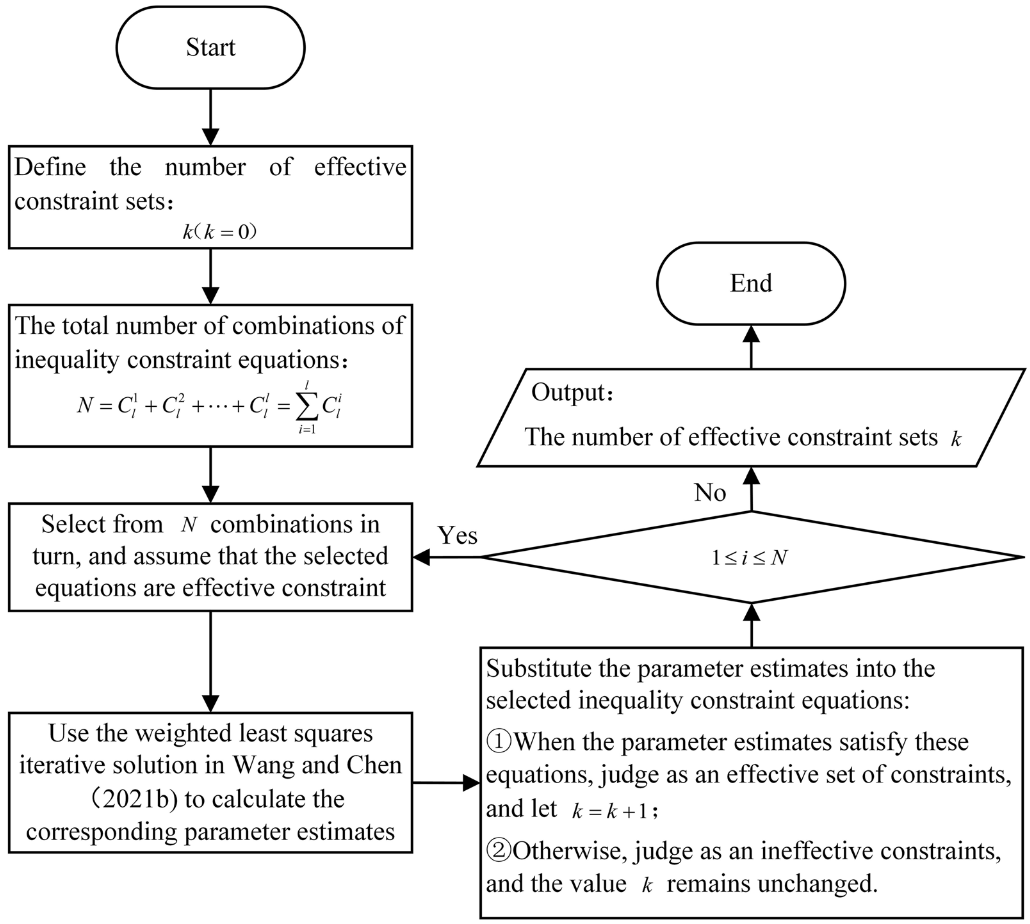

Fig. 1 Flowchart of utilizing the exhaustive search method to solve a MAAMRE model with inequality constraints

parameter estimation problem for a MAAMRE model with inequality constraints. Although the exhaustive search method is more computationally intensive when solving a problem with multiple inequality constraint equations, computational efficiency is no longer a problem with the rapid development of modern computers.

Ill-posed mixed additive and multiplicative random error model with inequality constraints random error model and its solution

Weighted least squares iterative regularization solution for an ill-posed mixed additive and multiplicative random error model

In the geodetic data processing field, ill-posed problems are widespread, and common solution methods include the ridge estimation method, the truncated singular value method, the conjugate gradient method and so on (Yang and Zhang 2010). At present, there are many studies on the parameter estimation problems for ill-posed GM, EIV and PEIV models (Wang 2006; Wang and Yu 2014; Wang et al. 2019), but there are few studies on ill-posed MAAMRE models, especially the parameter estimation problem for an ill-posed MAAMRE model with inequality constraints. Therefore, more in-depth research is needed. According to the idea of the exhaustive search approach, the ill-posed MAAMRE model with no constraints needs to be solved first before the corresponding ill-posed MAAMRE model with inequality constraints can be solved. In this paper, we adopt an improved Tikhonov regularization method to effectively reduce or eliminate the ill-posedness of the model to obtain a stable parameter solution after analyzing the causes of ill-posedness in MAAMRE models based on mathematical principles. In accordance with the Tikhonov regularization principle (Tikhonov and Arsenin 1977) and the Tikhonov regularization criterion for an ill-posed MRE model defined by Wang et al. (2021) and Zhao et al. (2022), this paper introduces a regularization factor α to construct the following regularization criterion for an ill-posed MAAMRE model:

$$\Phi = \boldsymbol{e}^{\mathrm{T}} \boldsymbol{Q}(\boldsymbol{e})^{-1} \boldsymbol{e} + \alpha \boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{\beta} = \min \tag{12}$$

According to the principle of the Lagrangian extreme value, after setting the partial derivatives of the target function Eq. (12) with respect to $\boldsymbol{\beta}$ equal to zero and transposing, one can obtain

$$\frac{\partial \Phi}{\partial \boldsymbol{\beta}} = 2\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{e})^{-1}\boldsymbol{e} + \boldsymbol{e}^{\mathrm{T}}\frac{\partial \boldsymbol{Q}(\boldsymbol{e})^{-1}}{\partial \boldsymbol{\beta}}\boldsymbol{e} + 2\alpha\boldsymbol{\beta} = \boldsymbol{0} \quad (13)$$

where $\boldsymbol{e}^{\mathrm{T}}\frac{\partial \boldsymbol{Q}(\boldsymbol{e})^{-1}}{\partial \boldsymbol{\beta}}\boldsymbol{e}$ represents the product of the residual of observation residuals, which is a very small value and can be neglected in the following calculation process. Thus, after a series of formula derivations, one can obtain

$$\hat{\boldsymbol{\beta}}_{\mathrm{RWLS}} = (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{e})^{-1}\boldsymbol{A} + \alpha\boldsymbol{I}_t)^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{e})^{-1}\boldsymbol{y} \quad (14)$$

where $\hat{\boldsymbol{\beta}}_{\mathrm{RWLS}}$ represents the weighted least squares regularization solution, α represents the regularization parameter, and $\boldsymbol{I}_t$ represents the tth-order unit matrix.

As seen from Eqs. (6) and (14), the weight matrix $\boldsymbol{Q}(\boldsymbol{e})^{-1}$ contains the parameter estimate $\hat{\boldsymbol{\beta}}_{\mathrm{RWLS}}$; thus, Eq. (14) needs to be solved iteratively and can be expressed as follows:

$$\hat{\boldsymbol{\beta}}^{i+1}_{\mathrm{RWLS}} = (\boldsymbol{A}^{\mathrm{T}}[\boldsymbol{Q}(\boldsymbol{e})^{i}]^{-1}\boldsymbol{A} + \alpha^{i}\boldsymbol{I}_t)^{-1}\boldsymbol{A}^{\mathrm{T}}[\boldsymbol{Q}(\boldsymbol{e})^{i}]^{-1}\boldsymbol{y} \quad (15)$$

where $\hat{\boldsymbol{\beta}}^{i+1}_{\mathrm{RWLS}}$ represents the weighted least squares iterative regularization solution and α^i represents the regularization parameter in the ith iteration.

The weighted least squares iterative regularization solution for an ill-posed MAAMRE model presented in this paper is summarized as Algorithm 2. The specific steps are as follows:

1) The observation $\boldsymbol{y}$ and the coefficient matrix $\boldsymbol{A}$ are taken as input and used to calculate the least squares solution as the initialization of the iterative process:

$$\hat{\boldsymbol{\beta}}^{0} = (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \quad (16)$$

2) The weight matrix $[\boldsymbol{Q}(\boldsymbol{e})^{i}]^{-1}$ for the ith iteration is calculated:

$$\boldsymbol{M}^{i}_{ma} = diag(\boldsymbol{A}\hat{\boldsymbol{\beta}}^{i}), \boldsymbol{D}(\boldsymbol{y})^{i} = \boldsymbol{M}^{i}_{ma}\boldsymbol{D}_m(\boldsymbol{M}^{i}_{ma})^{\mathrm{T}} + \boldsymbol{D}_a \quad (17)$$

$$[\boldsymbol{Q}(\boldsymbol{e})^{i}]^{-1} = \left(\frac{\boldsymbol{D}(\boldsymbol{y})^{i}}{\sigma_0^2}\right)^{-1} \quad (18)$$

3) The regularization parameter α^i for the ith iteration is calculated.
4) The weighted least squares iterative regularization solution $\hat{\boldsymbol{\beta}}^{i+1}_{\mathrm{RWLS}}$ is calculated via Eq. (15).
5) Steps 2)–4) are repeated until $||\hat{\boldsymbol{\beta}}^{i+1}_{\mathrm{RWLS}} - \hat{\boldsymbol{\beta}}^{i}_{\mathrm{RWLS}}|| \le 10^{-6}$ is satisfied, at which time the final parameter estimate $\hat{\boldsymbol{\beta}}^{i+1}_{\mathrm{RWLS}}$ is output.

According to the above steps, the flowchart of Algorithm 2 can be obtained, which is shown in Fig. 2.

Determination of the regularization parameter α

At present, there are many methods available to determine the regularization parameter, such as the generalized cross-validation (GCV) method (Golub et al. 1979), the ridge trace method (Hoerl and Kennard 1970), the L-curve method (Hansen 1992), the variance component estimation method (Wang et al. 2019) and so on. Different methods will yield different regularization parameters, and different regularization parameters will lead to different results. Therefore, it is necessary to choose an optimal method to determine the regularization parameter. Among the above methods, the GCV method cannot determine the optimal regularization parameter when the curve is too flat (Hansen 1992); the ridge trace method has certain subjective randomness when determining the regularization parameter; the L-curve method yields a graph from which the size of the regularization parameter cannot be directly read, and the solution result is too dependent on the curve fitting accuracy and may be nonconvergent (Xu 2010); and the variance component estimation method uses the regularization parameter as a virtual observation weight and may produce a negative variance (Wang and Wen 2017). Therefore, considering the above problems with the existing methods, this paper proposes the use of the U-curve method (Krawczyk-Stando and Rudnicki 2007; Wang et al. 2018; Wang and Chen 2021b) to determine the regularization parameter for an ill-posed MAAMRE model. The U-curve method uses a curvature formula to determine the regularization parameter, with the value of the maximum curvature point corresponding to the determined regularization parameter. The U-curve method does not require subjective or artificial determination decisions and can yield an accurate regularization parameter; thus, it can be better used to determine the regularization parameter for an ill-posed MAAMRE model.

Utilizing an exhaustive search method to solve an ill-posed mixed additive and multiplicative random error model with inequality constraints

Once the parameter solution for an ill-posed MAAMRE model with no constraints and with equality constrains has been obtained, a parameter estimation algorithm for an ill-posed MAAMRE model with inequality constraints

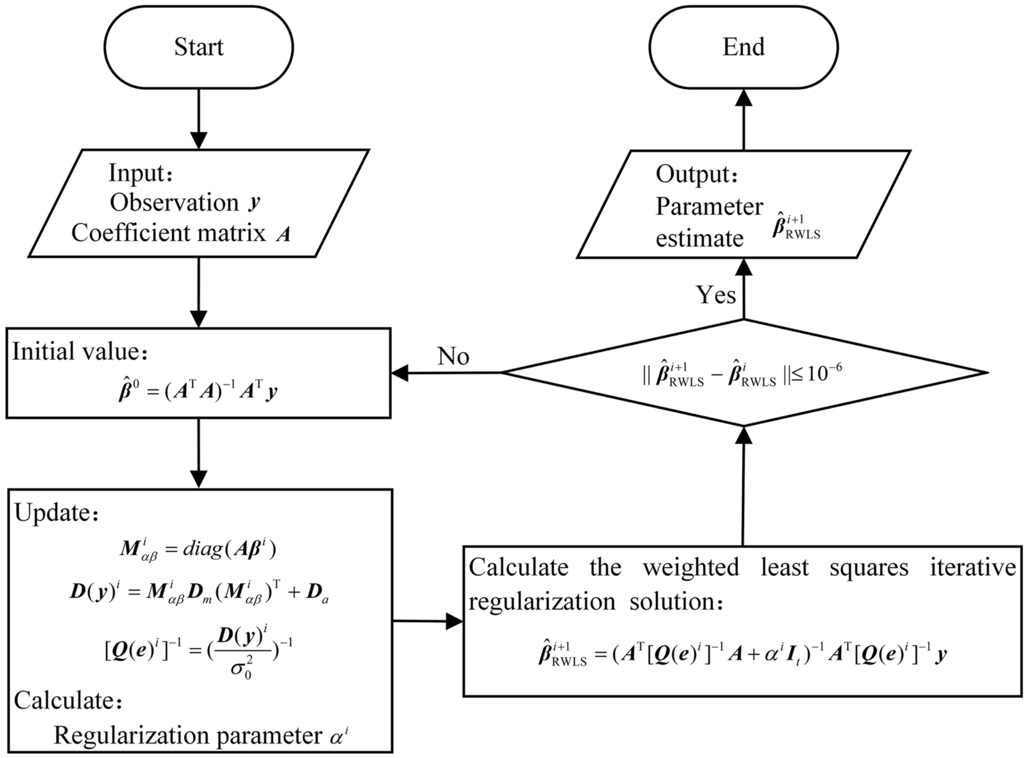

Fig. 2 Flowchart of the weighted least squares iterative solution for an ill-posed MAAMRE model

can be further established based on the idea of the exhaustive search method; in this paper, this algorithm is summarized as Algorithm 3. The flowchart of Algorithm 3 is shown in Fig. 3.

Similar to the previous discussion, we can obtain the following three conclusions regarding the value of kk:

1) When $kk = 0$ is satisfied, all sets of inequality constraint equations are all ineffective constraints. At this time, the optimal parameter solution $\hat{\boldsymbol{\beta}}$ for the ill-posed MAAMRE model with inequality constraints is equal to the parameter solution for the unconstrained ill-posed MAAMRE model, which is also the weighted least squares iterative regularization solution proposed in Algorithm 2.
2) When $kk = 1$ is satisfied, there exists one set of effective constraint equations. At this time, the optimal parameter solution $\hat{\boldsymbol{\beta}}$ for the ill-posed MAAMRE model with inequality constraints is equal to the parameter solution obtained based on this set of effective constraint equations.
3) When $kk > 1$ is satisfied, multiple sets of effective constraint equations exist. At this time, there are kk sets of parameter solutions, and we need to select the optimal parameter solution $\hat{\boldsymbol{\beta}}$ for the ill-posed MAAMRE model with inequality constraints from among them. Similar to the method in Sect. 3, considering that the kk sets of parameter solutions are all obtained as the ridge estimation iterative solution, in this paper, the kk sets of parameter solutions are all substituted into the weighted square sum of the residuals in turn, and the parameter solution corresponding to the minimum value is selected as the optimal parameter solution for the ill-posed MAAMRE model with inequality constraints.

Examples and analysis

To verify the performance of the proposed algorithms in solving the parameter estimation problem for a MAAMRE model with inequality constraints and to facilitate a discussion of their application value in geodesy, this paper presents two sets of examples designed for verification and analysis: global positioning system (GPS) elevation fitting model examples and DTM model examples. To better highlight the effectiveness and advantages of the proposed algorithms, the existing least squares solution, weighted least squares iterative solution and bias-corrected weighted least squares iterative solution for MAAMRE models given in Xu et al. (2013) are used for comparison. For convenience, the methods

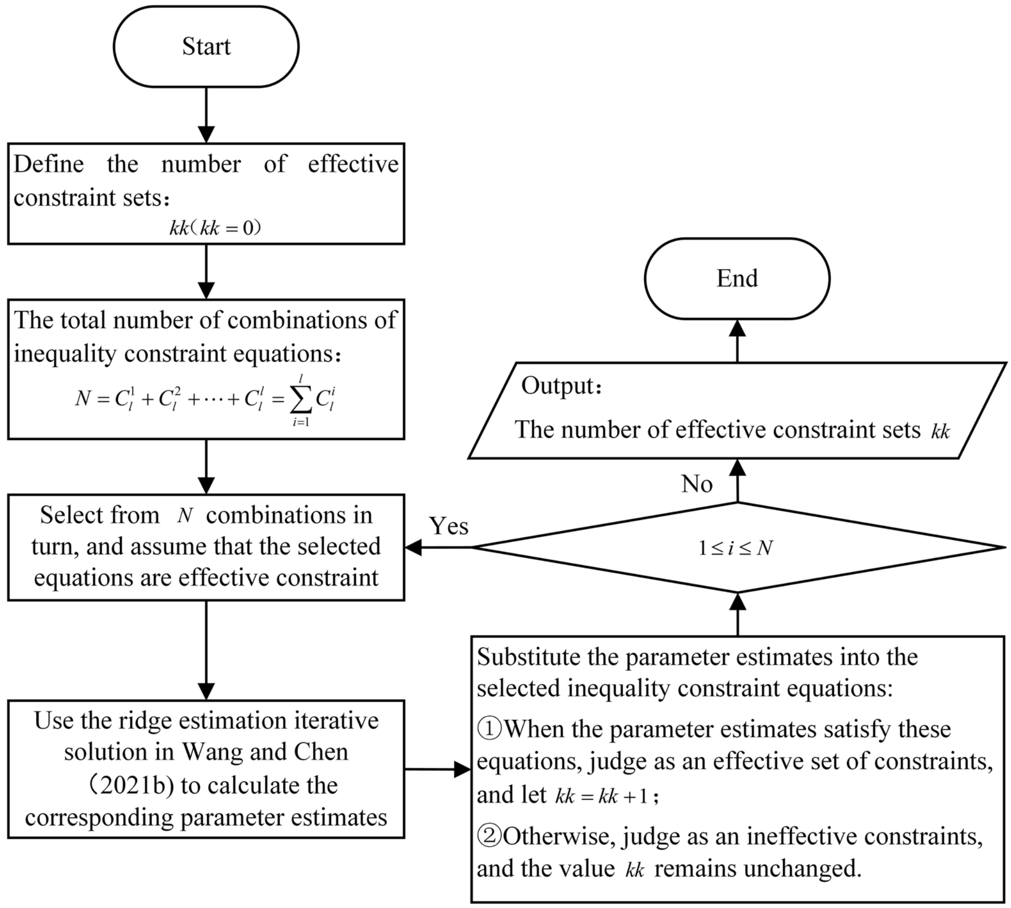

Fig. 3 Flowchart of utilizing the exhaustive search method to solve an ill-posed MAAMRE model with inequality constraints

Table 1 Six schemes and their corresponding methods

Scheme	Method
1	Least squares solution (LS)
2	Weighted least squares iterative solution (WLS)
3	Bias-corrected weighted least squares iterative solution (bcWLS)
4	Algorithm 1
5	Algorithm 2
6	Algorithm 3

corresponding to each of the six schemes referred to in the examples below are listed in Table 1.

GPS elevation fitting model

To preliminarily verify the feasibility and advantages of the proposed algorithms, Examples 1 and 2 will use a GPS elevation fitting model example for verification and analysis. Since the two inequality constraint algorithms proposed in this paper are derived for the cases of non-ill-posed and ill-posed coefficient matrices, Examples 1 and 2 are based on simulations of non-ill-posed and ill-posed GPS elevation fitting problems disturbed by MAAMREs, respectively.

Example 1

Example 1 is based on the observation equation for a non-ill-posed GPS elevation fitting model given in Chen (2017), and the function model is as follows:

$$y = 10 + x + 2x^2 + 2x^3 \tag{19}$$

where x represents the distance between the elevations of a ground points and the origin of the coordinate system divided by 100, with a value range of 0–300 m in intervals of 10 m, y represents the true value of the ground elevation point obtained based on the value x in Eq. (19), and the parameters to be estimated are the five coefficients of Eq. (19), namely, $\tilde{\boldsymbol{\beta}} = \begin{bmatrix} 10 & 1 & 2 & 2 \end{bmatrix}^{\mathrm{T}}$.

Example 2

Example 2 is based on the observation equation for an ill-posed GPS elevation fitting model given in Wang and Chen (2021a), and the function model is as follows:

$$y = 10 + x + x^2 + x^3 + 0.2x^4 \tag{20}$$

where x represents the distance between the elevations of a ground point and the origin of the coordinate system divided by 100, with a value range is 0–400 m in intervals of 10 m, y represents the true value of the ground elevation point obtained based on the value x in Eq. (20), and the parameters to be estimated are the five coefficients of Eq. (20), namely, $\tilde{\boldsymbol{\beta}} = \begin{bmatrix} 10 & 1 & 1 & 1 & 0.2 \end{bmatrix}^{\mathrm{T}}$.

Result analysis

According to Wang and Chen (2021a), the corresponding observation equation for a GPS elevation fitting model disturbed by MAAMREs can be expressed as follows (Wang and Chen 2021a):

$$\boldsymbol{Y} = \boldsymbol{y} \odot (\mathbf{1} + \boldsymbol{\varepsilon}_m) + \boldsymbol{\varepsilon}_a \tag{21}$$

where $\boldsymbol{Y}$ represents the observation vector of ground elevation points disturbed by MAAMREs, $\boldsymbol{y}$ represents the vector of the true values of the ground elevation points, and 1 represents a unit column vector.

As can be seen from Eq. (21), the MAAMREs affects the observations in both additive and multiplicative forms, which is clearly inconsistent with traditional theories and methods; thus, further research is needed. In accordance with Wang and Chen (2021a), for this example, the standard deviations of the multiplicative and additive random errors are set to 0.05 and 0.15 m, respectively, and the random errors are generated by the mvnrnd function in MATLAB. To illustrate the extent to which the elevations of the ground elevation points are affected by the random errors, the elevations before and after disturbance by random errors are plotted in Figs. 4 and 5, respectively. Then, following the practice of prior inequality constraint conditions between the parameters of a linear equation system defined in Xie (2014), the prior inequality constraint conditions shown in Tables 2 and 3 are added in Examples 1 and 2, respectively. At this time, the physical meaning of the inequality constraint conditions can be expressed as a prior conditions that must be satisfied by the parameter estimates. In actual data processing, these prior conditions may be derived from previous observations or the requirements to be satisfied in the practical situation.

In Figs. 4 and 5, the curves and circles represent the true values and observations, respectively, of the elevations of the ground elevation points. The circles greatly deviate from the curves, indicating that the elevations have suffered severe deviations due to the effect of the MAAMREs. Therefore, further research is needed regarding a GPS elevation fitting model disturbed by MAAMREs.

The unit weight error σ_0 is set to 0.3 (Wang and Chen 2021a). For Example 1, the LS method, the WLS method, the bcWLS method and Algorithm 1 are used for calculation, and for Example 2, the LS method, the

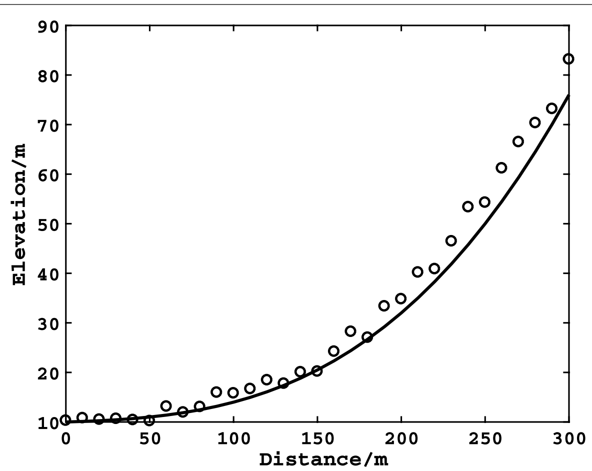

Fig. 4 Elevations of the ground elevation points before and after disturbance by MAAMREs in the first example

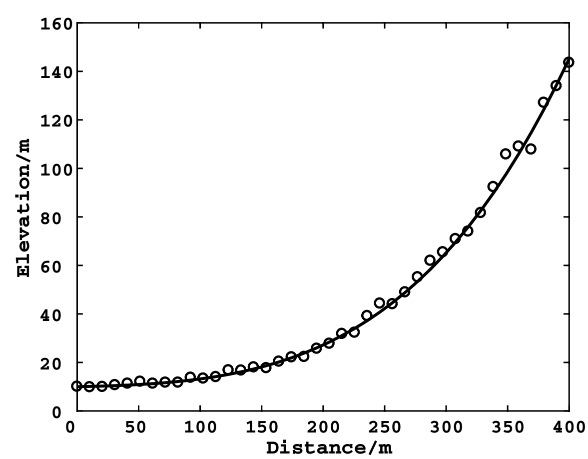

Fig. 5 Elevations of the ground elevation points before and after disturbance by MAAMREs in the second example

Table 2 Inequality constraint data for the first example

G				*W*
1	1	0	0	11
0	1	1	0	3
0	0	1	1	4

Table 3 Inequality constraint data for the second example

G						*W*
I	1	1	0	0	0	11
	0	1	1	0	0	3
	0	0	1	1	0	3
	0	0	0	1	1	1.2
II	1	1	0	0	0	11
	0	1	1	0	0	2
	0	0	1	1	0	2
	0	0	0	1	1	1.2

Table 4 Parameter estimates and the 2-norm results between the parameter estimates and true values for the first example

Parameter	LS	WLS	bcWLS	Algorithm 1	True value
$\hat{\beta}_1$	10.5236	10.2059	10.2112	10.1353	10
$\hat{\beta}_2$	− 0.3301	1.3997	1.2223	0.8647	1
$\hat{\beta}_3$	3.0391	1.5186	1.6319	2.1353	2
$\hat{\beta}_4$	1.6868	2.0327	2.0121	1.8647	2
$\left\|\Delta\hat{\boldsymbol{\beta}}\right\|$	1.7948	0.6595	0.4793	0.2705	/

WLS method, the bcWLS method and Algorithms 2 and 3 are used for calculation. The parameter estimates $\hat{\boldsymbol{\beta}}$ and the 2-norms $\left\|\Delta\hat{\boldsymbol{\beta}}\right\|$ between the parameter estimates and the true values are listed in Tables 4 and 5, respectively.

As can be seen from the results in Table 4, the parameter estimates obtained with Algorithm 1 are closer to the true values than the three existing solutions given by Xu et al. (2013), indicating that considering the prior inequality constraint information during adjustment can further improve the results of parameter estimation and that applying the exhaustive search method to solve the MAAMRE model with inequality constraints is feasible and effective. Meanwhile, the parameter estimates obtained with Algorithm 1 and the inequality constraint data show that the three constraints in the optimal solution are all effective constraints.

By comparing the results of the LS method, the WLS method, the bcWLS method and Algorithms 2 and 3 in Table 5, it can be seen that the parameter estimates obtained with the three existing solutions given by Xu et al. (2013) seriously deviate from the true values; this is because when deriving the formulas for the three existing solutions, the ill-posed coefficient matrix was not considered. Meanwhile, by comparing the results of Algorithms 2 and 3, it can be seen that the parameter estimates obtained with Algorithm 3 are closer to the true values than those of Algorithm 2, which further shows the effectiveness and advantages of using the exhaustive search method to solve the ill-posed MAAMRE model with inequality constraints. In addition, the parameter estimates obtained with Algorithm 3 and the inequality constraint data show that the four constraints in the two sets of optimal solutions are all effective constraints. However, since the second set of inequality constraint conditions is closer to the true values than the first set, the parameter estimates obtained are also closer to the true values.

Digital terrain model

In actual data processing, a model may easily be ill-posed; thus, Example 3 considers the parameter estimation problem for an ill-posed DTM model with inequality

Table 5 Parameter estimates and the 2-norm results between the parameter estimates and true values for the second example

Parameter	LS	WLS	bcWLS	Algorithm 2	Algorithm 3 I	Algorithm 3 II	True value
$\hat{\beta}_1$	9.4812	9.7034	9.6968	7.6056	9.8685	9.9992	10
$\hat{\beta}_2$	5.6544	3.9896	3.9633	2.6792	1.1315	1.0008	1
$\hat{\beta}_3$	− 5.2673	− 2.7707	− 2.7753	1.3940	1.8685	0.9992	1
$\hat{\beta}_4$	3.8171	2.6352	2.6423	0.6744	1.1315	1.0008	1
$\hat{\beta}_5$	− 0.1891	− 0.0181	− 0.0199	0.2426	0.0685	0.1992	0.2
$\left\|\Delta\hat{\boldsymbol{\beta}}\right\|$	8.3246	5.0957	5.0864	2.9692	0.9074	0.0019	/

constraints. DTMs, which realize the general expression of natural surface morphologies in numerical form, have been widely used in many information engineering constructions (Shekhar and Xiong 2008; He 2014). In this example, we will imitate the practice in Wang and Chen (2021a, 2021b) and use the interpolation method to simulate the observation equation for an ill-posed DTM model disturbed by MAAMREs, which can be expressed as follows:

$$\boldsymbol{H}(x,y) = \boldsymbol{h}(x,y) \odot (\mathbf{1} + \boldsymbol{\varepsilon}_m) + \boldsymbol{\varepsilon}_a \tag{22}$$

where $\boldsymbol{H}(x,y) \in \boldsymbol{R}^{1681\times 1}$ represents the observation vector of the DTM model elevations disturbed by MAAMREs, $\mathbf{1} \in \boldsymbol{R}^{1681\times 1}$ represents a unit column vector, and $\boldsymbol{h}(x,y) \in \boldsymbol{R}^{1681\times 1}$ represents the vector of the true values of the DTM model elevations, expressed as $\boldsymbol{h}(x,y) = \sum_{i=1}^{4} \beta_i f_i(x,y)$. The true values of the parameter estimates are $\tilde{\boldsymbol{\beta}} = [1.52010 - 4]^{\mathrm{T}}$, and the functions $f_i(x,y)(i = 1,2,3,4)$ are as follows (Wang and Chen 2021a, 2021b):

$$\begin{cases} f_1(x,y) = \exp\left\{-[(x-22)^2 + (y-22)^2]/500\right\} \\ f_2(x,y) = \exp\left\{-[(x-28)^2 + (y-28)^2]/500\right\} \\ f_3(x,y) = \exp\left\{-[(x-25)^2 + (y-25)^2]/500\right\} \\ f_4(x,y) = \exp\left\{-[(x-20)^2 + (y-20)^2]/500\right\} \end{cases} \tag{23}$$

where the value ranges of the ground points on both the abscissa x and the ordinate y are 0–80 m with intervals of 2 m.

At present, observations that exhibit MAAMREs have not received sufficient attention, and existing research on MAAMREs does not give the specific standard deviations of the additive and multiplicative random errors. Therefore, following the practice of the existing studies on MAAMRE models (Xu et al. 2013; Shi 2014; Shi and Xu 2021; Wang and Chen 2021a, 2021b), in this example, the standard deviations of the MREs are set to 0.005, 0.05 and 0.1, and the standard deviation of the AREs is set to 0.15 m, based on the theoretical and practical tests on LiDAR data reported by Flamant et al. (1984), Wang and Pruitt (1992), Kraus and Pfeifer (1998), Berg and Ferguson (2000), Hill et al. (2003), Kobler et al. (2007), Leigh et al. (2010) and Veneziano et al. (2010). Meanwhile, to illustrate the extent to which the DTMs are affected by MAAMREs, the DTMs before and after disturbance by random errors are plotted in Figs. 6, 7, 8 and 9.

By comparing the DTM model elevations in Figs. 6, 7, 8 and 9, it can be seen that the elevations in Fig. 6 are smooth, while the elevations in Figs. 7, 8 and 9 exhibit

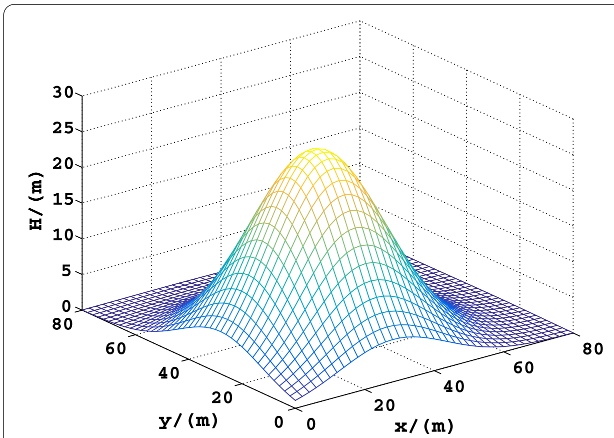

Fig. 6 DTM model without random errors in the third example

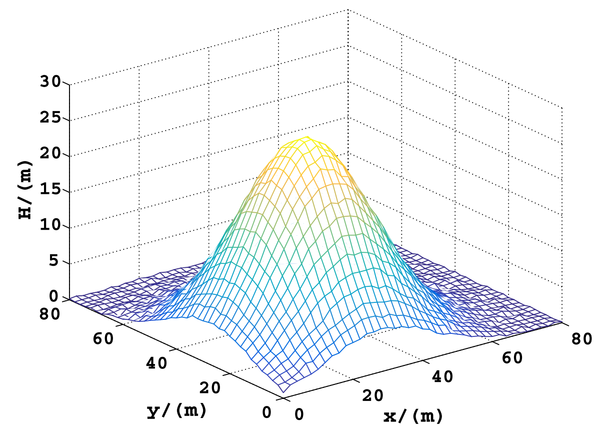

Fig. 7 DTM model disturbed by MREs with $\sigma_m = 0.005$ in the third example

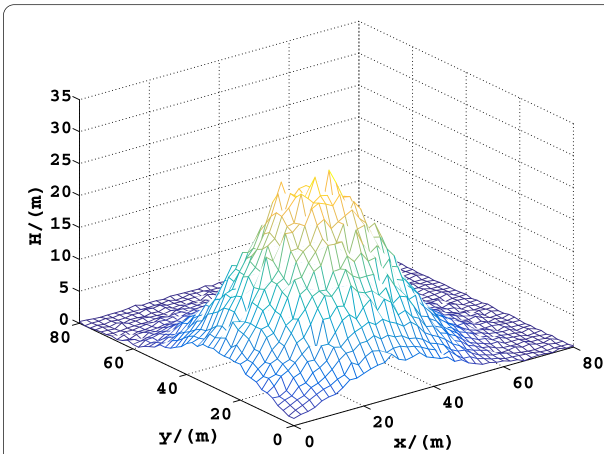

Fig. 8 DTM model disturbed by MREs with $\sigma_m = 0.05$ in the third example

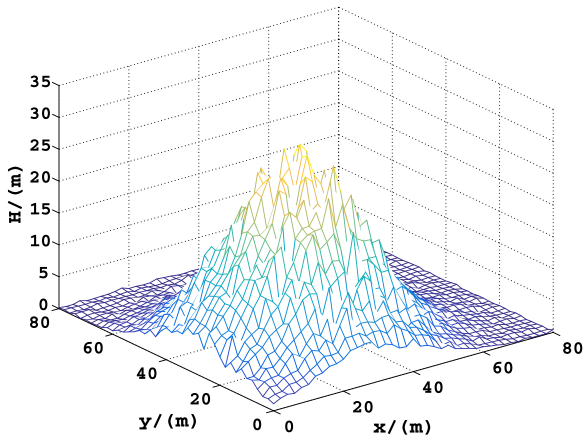

Fig. 9 DTM model disturbed by MREs with $\sigma_m = 0.1$ in the third example

folds that become increasingly severe as the MRE increases, indicating that the random errors have a greater impact on the DTM model elevations. After calculation, the condition number of the normal equation matrix is 7.1054×10^5, indicating that the model is severely ill-posed. Therefore, detailed research is needed on ill-posed DTMs disturbed by MAAMREs.

To further explain the influence of the range of the inequality constraint conditions on parameter estimation, two sets of inequality constraint conditions are designed for this example, as shown in Table 6. For the case in which the prior unit weight error σ_0 is set to 0.3 (Xu et al. 2013; Shi 2014; Shi and Xu 2021; Wang and Chen 2021a, 2021b) and the first set of inequality constraint conditions is used for calculation, the results of the parameter estimates and the 2-norms between the parameter estimates and the true values are listed in Table 7. As seen from the inequality constraint data in Table 6, the second set of inequality constraint conditions is closer to the true values than the first set; therefore, the parameter estimates obtained using the second set of inequality constraint conditions also should theoretically be closer to the true values. For further verification, the results obtained in the case of $\sigma_m = 0.05$ when the second set of inequality constraint conditions is used for calculation are listed in Table 8.

As seen from the 2-norm results in Table 7, the parameter estimates obtained with the three existing solutions given by Xu et al. (2013) greatly deviate from the true values. Meanwhile, by comparing the results for

Table 6 Inequality constraint data for the third example

G					W
I	1	1	1	1	28
	1	0	0	0	1.6
	0	1	0	0	20.1
	0	0	1	0	10.1
II	1	1	1	1	27.5
	1	0	0	0	1.5
	0	1	0	0	20
	0	0	1	0	10

Table 7 Parameter estimates and the 2-norm results between the parameter estimates and true values for the third example

Parameter		LS	WLS	bcWLS	Algorithm 2	Algorithm 3 I	True value
$\sigma_m = 0.005$	$\hat{\beta}_1$	4.2871	3.0437	3.0323	1.4492	1.6000	1.5
	$\hat{\beta}_2$	20.4395	20.1661	20.1635	19.6415	19.8517	20
	$\hat{\beta}_3$	8.2360	9.1576	9.1653	10.4929	10.1000	10
	$\hat{\beta}_4$	−5.4799	−4.8835	−4.8778	−4.1076	−4.0686	−4
	$\left\Vert\Delta\hat{\beta}\right\Vert$	3.6419	1.9751	1.9601	0.6209	0.2161	/
$\sigma_m = 0.05$	$\hat{\beta}_1$	17.4372	10.6262	9.8311	1.6658	1.6000	1.5
	$\hat{\beta}_2$	23.1451	21.8015	21.5658	19.2789	20.1000	20
	$\hat{\beta}_3$	−1.1115	3.8043	4.3941	10.4287	10.1000	10
	$\hat{\beta}_4$	−12.0305	−8.7000	−8.3279	−4.0328	−4.2751	−4
	$\left\Vert\Delta\hat{\beta}\right\Vert$	21.2566	12.1247	11.0461	0.8557	0.3251	/
$\sigma_m = 0.1$	$\hat{\beta}_1$	31.8167	14.5611	13.6686	1.6070	1.6000	1.5
	$\hat{\beta}_2$	26.2958	24.3009	23.8356	19.1696	20.1000	20
	$\hat{\beta}_3$	−12.2376	−1.7409	−1.0338	9.7767	10.1000	10
	$\hat{\beta}_4$	−18.4205	−1.7409	−8.9716	−3.2539	−4.1213	−4
	$\left\Vert\Delta\hat{\beta}\right\Vert$	40.7579	18.8592	17.5854	1.1435	0.2115	/

Table 8 Parameter estimates and the 2-norm results between the parameter estimates and true values in the case of $\sigma_m = 0.05$ for the third example

Parameter	LS	WLS	bcWLS	Algorithm 2	Algorithm 3 I	Algorithm 3 II	True value
$\hat{\beta}_1$	17.4372	10.6262	9.8311	1.6658	1.6000	1.5000	1.5
$\hat{\beta}_2$	23.1451	21.8015	21.5658	19.2789	20.1000	20.0000	20
$\hat{\beta}_3$	− 1.1115	3.8043	4.3941	10.4287	10.1000	10.0000	10
$\hat{\beta}_4$	− 12.0305	− 8.7000	− 8.3279	− 4.0328	− 4.2751	− 4.0358	− 4
$\left\|\Delta\hat{\beta}\right\|$	21.2566	12.1247	11.0461	0.8557	0.3251	0.0358	/

the three sets of MREs, it can be seen that the degree of elevation deviation gradually increases as the MRE becomes larger. Since the ill-posed nature of the coefficient matrix was considered when deriving the formulas for the proposed algorithms, the parameter estimates obtained are closer to the true values. Among them, the 2-norm results obtained with Algorithm 2 are reduced by 68.32%, 92.25%, and 93.5% compared to the results of the bcWLS method, and the 2-norm results obtained with Algorithm 3 are reduced by 65.2%, 62.01% and 81.5% compared to the results of Algorithm 2, indicating that the proposed algorithms are affected by the size of the MRE; specifically, the greater the MRE is, the more obvious the advantages of the proposed algorithms. In addition, as seen from the parameter estimates obtained with Algorithm 3 in Table 8 and the inequality constraint data in Table 6, the four constraints in both sets of optimal solutions are all effective constraints. However, since the second set of inequality constraint conditions is closer to the true values than the first set, the parameter estimates obtained with the second set of inequality constraint conditions are closer to the true values. This finding is consistent with the conclusion obtained in Example 2 and further illustrates the feasibility and advantages of the proposed algorithms.

Conclusions

The theories and methods for inequality constraint models based on AREs cannot satisfy the needs of MAAMRE models with inequality constraints; thus, it is important to propose corresponding parameter estimation theories and methods.

In this paper, we first explain and analyze the basic formulas and statistical properties of MAAMRE models. Then, a MAAMRE model with inequality constraints is established based on the definition of inequality constraint equations. In addition, by applying the ideas of active and inactive constraints and the exhaustive search approach, parameter estimation algorithms for a MAAMRE model with inequality constraints and an ill-posed MAAMRE model with inequality constraints are proposed. Finally, through the comparative analysis of various methods on several examples, we conclude that the proposed Algorithm 1 can yield obtain more accurate parameter estimates than three existing solutions, indicating that considering the prior inequality constraint information during the adjustment process can further improve the quality of parameter estimates and that using the exhaustive search method to solve the parameter estimation problem for a MAAMRE model with inequality constraints is feasible and effective. In addition, when the MAAMRE model is ill-posed, Algorithm 2 can obtain reasonable parameter estimates, while Algorithm 3 can obtain more accurate parameter estimates. However, the prior inequality constraint conditions will affect the parameter estimates obtained; in particular, when the range of the prior conditions is closer to the true values, the parameter estimates obtained will also be closer to the true values.

The algorithms proposed in this paper are an important supplement to the literature on the parameter estimation problem for MAAMRE models with inequality constraints; however, they are applicable only to the case in which the inequality constraint matrix is of full-rank. Hence, the next focus of related research should be the rank-deficit case.

Abbreviations

ARE: Additive random error; MAAMRE: Mixed additive and multiplicative random error; GPS: Global positioning system; DTM: Digital terrain model; MRE: Multiplicative random error; SAR: Synthetic aperture radar; LiDAR: Light detection and ranging; GNSS: Global navigation satellite system; VLBI: Very long baseline interferometry; EDM: Electronic distance measurement; GM: Gauss–Markov; EIV: Errors-in-variables; PEIV: Partial errors-in-variables; GCV: Generalized cross validation; LS: Least squares solution; WLS: Weighted least squares iterative solution; bcWLS: Bias-corrected weighted least squares iterative solution.

Acknowledgements

The authors are grateful to all of the anonymous reviewers and Editor-in-Chief Prof. Takeshi Sagiya for their careful review and valuable suggestions, which improved the quality of this paper.

Author contributions

All authors contributed significantly to the manuscript. LW initiated the idea, provided critical comments and contributed to the final revision of the paper. TC performed all data processing and analyses and contributed to drafting the manuscript. All authors read and approved the final manuscript.

Funding
This research is supported by the National Natural Science Foundation of China, Grant Nos. 42174011 and 41874001, and by the Innovation Found Designated for Graduate Students of ECUT, Grant No. DHYC-202020.

Availability of data and materials
The data sets are available from the corresponding author upon reasonable request.

Declarations

Competing interests
The authors declare that they have no competing interests.

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Received: 10 March 2022 Accepted: 30 July 2022
Published online: 13 August 2022

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