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A correction to Bahr's "phase deviation" method for tensor decomposition

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For models having moderate departures from the basic distortion model (from the so called "principal superimposition" model, that is where local three-dimensional and regional two-dimensional structures are superimposed) a special tensor decomposition method (the so called phase deviation method) was suggested by Bahr (1991). As far as we know, this technique has never got a wide field application in the field. In a careful examination of the suggested solution, an error was observed in the original derivation of the formulas. In this paper Bahr's (1991) solution is corrected. Using the new equations, more understandable and interpretable results are obtained, as it is illustrated on synthetic examples.

1. Introduction

For the interpretation of a measured impedance tensor—according to Bahr (1988)—one must ask, whether all elements of the measured tensor have the same phase. If they do, the regional conductivity is purely depth dependent and it is sufficient to split the impedance tensor into a real distortion matrix and a scalar normal impedance.

If only the two elements in each column of the measured tensor have equal phase values, the regional conductivity is two-dimensional. The tensor decomposition solution for this so called principal superimposition model is found in the paper by Bahr (1988).

If the regional conductivity distribution is not perfectly two-dimensional, a phase difference will appear between the two elements of the same column. For this problem—assuming a moderate departure from the principal superimposition model—a solution was given by Bahr (1991), which he calls the "phase deviation" method. We found an error in the original derivation of the formulas. In this paper we first give a brief description of the phase deviation method, then we present the corrected solution. Finally a comparison between the original and the corrected solution is given.

2. Brief Description of Bahr's (1991) Phase Deviation Method

In case of moderate departures from the principal superimposition model, Bahr (1991) represented the measured magnetotelluric tensor $\mathbf{Z}$ in the coordinate system of the regional 2D structure as

$$\mathbf{Z} = \begin{bmatrix} -a_{12} Z_{TM} e^{i\delta} & a_{11} Z_{TE} \\ -a_{22} Z_{TM} & a_{21} Z_{TE} e^{-i\delta} \end{bmatrix}, \tag{1}$$

where $\mathbf{a} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a real, frequency-independent matrix, representing the distortion due to small, localized, near-surface anomaly. Z_{TE} and Z_{TM} are the principal impedances for the 2D regional structure. According to the condition set up by Bahr (1991), the phases between the elements in both columns of the tensor—due to the effect of the phase-sensitive regional skew η (Bahr, 1991)—differ by the same phase deviation angle δ.

The strike angle α is found from the two conditions for the two columns of the impedance tensor of Eq. (1), whereas the two variables α and δ are to be resolved. From a comparison of the real and imaginary parts of elements in both columns, two equation are obtained:

$$\frac{\operatorname{Re} Z_{x'x'} \cos\delta + \operatorname{Im} Z_{x'x'} \sin\delta}{\operatorname{Re} Z_{y'x'}} = \frac{-\operatorname{Re} Z_{x'x'} \sin\delta + \operatorname{Im} Z_{x'x'} \cos\delta}{\operatorname{Im} Z_{y'x'}}, \tag{2a}$$

$$\frac{\operatorname{Re} Z_{y'y'} \cos\delta - \operatorname{Im} Z_{y'y'} \sin\delta}{\operatorname{Re} Z_{x'y'}} = \frac{\operatorname{Re} Z_{y'y'} \sin\delta + \operatorname{Im} Z_{y'y'} \cos\delta}{\operatorname{Im} Z_{x'y'}}. \tag{2b}$$

In another form:

$$\cos\delta(\operatorname{Re} Z_{x'x'} \cdot \operatorname{Im} Z_{y'x'} - \operatorname{Re} Z_{y'x'} \cdot \operatorname{Im} Z_{x'x'}) + \sin\delta(\operatorname{Re} Z_{x'x'} \cdot \operatorname{Re} Z_{y'x'} + \operatorname{Im} Z_{x'x'} \cdot \operatorname{Im} Z_{y'x'}) = 0 \tag{3a}$$

and

$$\cos\delta(\operatorname{Re} Z_{y'y'} \cdot \operatorname{Im} Z_{x'y'} - \operatorname{Re} Z_{x'y'} \cdot \operatorname{Im} Z_{y'y'}) - \sin\delta(\operatorname{Re} Z_{x'y'} \cdot \operatorname{Re} Z_{y'y'} + \operatorname{Im} Z_{x'y'} \cdot \operatorname{Im} Z_{y'y'}) = 0. \tag{3b}$$

To solve these equations, Bahr (1991) introduced

(1) two commutators between the complex numbers C_1 and C_2 as follows:

$$[C_1, C_2] = \operatorname{Im}(C_2 C_1^*) = \operatorname{Re} C_1 \cdot \operatorname{Im} C_2 - \operatorname{Re} C_2 \cdot \operatorname{Im} C_1, \tag{4a}$$

$$\{C_1, C_2\} = \mathrm{Re}(C_2 C_1^*) = \mathrm{Re}\, C_1 \cdot \mathrm{Re}\, C_2 + \mathrm{Im}\, C_1 \cdot \mathrm{Im}\, C_2. \tag{4b}$$

In this way Eqs. (3a) and (3b) have the following simpler form:

$$\cos\delta[Z_{x'x'}, Z_{y'x'}] + \sin\delta\{Z_{x'x'}, Z_{y'x'}\} = 0, \tag{5a}$$
$$\cos\delta[Z_{y'y'}, Z_{x'y'}] - \sin\delta\{Z_{y'y'}, Z_{x'y'}\} = 0, \tag{5b}$$

and (2) he defined four so called modified impedances

$$S_1 = \frac{Z_{xx} + Z_{yy}}{2}, \tag{6a}$$
$$S_2 = \frac{Z_{xy} + Z_{yx}}{2}, \tag{6b}$$
$$D_1 = \frac{Z_{xx} - Z_{yy}}{2}, \tag{6c}$$
$$D_2 = \frac{Z_{xy} - Z_{yx}}{2}. \tag{6d}$$

Their transformation into a new coordinate system which is rotated by an angle α yields

$$S_1' = \frac{Z_{x'x'} + Z_{y'y'}}{2} = S_1, \tag{7a}$$
$$S_2' = \frac{Z_{x'y'} + Z_{y'x'}}{2} = S_2 \cos 2\alpha - D_1 \sin 2\alpha, \tag{7b}$$
$$D_1' = \frac{Z_{x'x'} - Z_{y'y'}}{2} = D_1 \cos 2\alpha + S_2 \sin 2\alpha, \tag{7c}$$
$$D_2' = \frac{Z_{x'y'} - Z_{y'x'}}{2} = D_2. \tag{7d}$$

Therefore $Z_{x'x'}$, $Z_{x'y'}$, $Z_{y'x'}$ and $Z_{y'y'}$ are a function of α:

$$Z_{x'x'} = S_1 + D_1' = S_1 + D_1 \cos 2\alpha + S_2 \sin 2\alpha, \tag{8a}$$
$$Z_{x'y'} = S_2' + D_2 = D_2 + S_2 \cos 2\alpha - D_1 \sin 2\alpha, \tag{8b}$$
$$Z_{y'x'} = S_2' - D_2 = -D_2 + S_2 \cos 2\alpha - D_1 \sin 2\alpha, \tag{8c}$$
$$Z_{y'y'} = S_1 - D_1' = S_1 - D_1 \cos 2\alpha - S_2 \sin 2\alpha. \tag{8d}$$

By using the commutators (4a and b) and the modified impedances (8a, b, c and d) Bahr (1991) expressed Eqs. (5a) and (5b) as:

$$-A \sin 2\alpha + B \cos 2\alpha + C + E \cos 2\alpha \cdot \sin 2\alpha = 0, \tag{9a}$$
$$-A^+ \sin 2\alpha + B^+ \cos 2\alpha + C^+ + E^+ \cos 2\alpha \cdot \sin 2\alpha = 0, \tag{9b}$$

where for the first equation (if indices 1 refer to the terms with $\cos\delta$ and indices 2 refer to those with $\sin\delta$)

$$A = A_1 + A_2 = ([S_1, D_1] + [S_2, D_2]) \cos\delta + (\{S_1, D_1\} + \{S_2, D_2\}) \sin\delta, \tag{10a}$$
$$B = B_1 + B_2 = ([S_1, S_2] - [D_1, D_2]) \cos\delta + (\{S_1, S_2\} - \{D_1, D_2\}) \sin\delta, \tag{10b}$$
$$C = C_1 + C_2 = ([D_1, S_2] - [S_1, D_2]) \cos\delta + (\{D_1, S_2\} - \{S_1, D_2\}) \sin\delta, \tag{10c}$$
$$E = E_2 = (\{S_1, S_1\} - \{D_2, D_2\}) \sin\delta, \tag{10d}$$

and for the second equation

$$A^+ = A_1 - A_2, \tag{11a}$$
$$B^+ = B_1 - B_2, \tag{11b}$$
$$C^+ = -C_1 + C_2, \tag{11c}$$
$$E^+ = E_2 = E. \tag{11d}$$

3. Problem

We have found that in Eqs. (9a) and (9b) a further term must exist. In addition, the indices in Eq. (10d) are mixed up and some other misprints should be corrected, too.

We think, the origin of the problem in the derivation by Bahr (1991) must be an erroneous transformation of the commutator $\{C_1, C_2\}$.

4. Correction

Since

$$C_2 C_1^* = a_2 e^{i\varphi_2} \cdot a_1 e^{-i\varphi_1} = a_1 a_2 \cos(\varphi_2 - \varphi_1) + i a_1 a_2 \sin(\varphi_2 - \varphi_1), \tag{12}$$

the commutative laws for the two commutators are:

$$\{C_1, C_2\} = \{C_2, C_1\}, \tag{13a}$$
$$[C_1, C_2] = -[C_2, C_1]. \tag{13b}$$

It is inevitable to express $\{Z_{x'x'}, Z_{y'x'}\}$ in details:

$$\begin{aligned}\{Z_{x'x'}, Z_{y'x'}\} &= \{S_1 + D_1 \cos 2\alpha + S_2 \sin 2\alpha, -D_2 - D_1 \sin 2\alpha + S_2 \cos 2\alpha\} \\ &= -\{S_1, D_2\} - \{D_1, D_2\} \cos 2\alpha - \{S_2, D_2\} \sin 2\alpha - \{S_1, D_1\} \sin 2\alpha \\ &\quad - \{D_1, D_1\} \sin 2\alpha \cdot \cos 2\alpha - \{S_2, D_1\} \sin^2 2\alpha + \{S_1, S_2\} \cos 2\alpha \\ &\quad + \{D_1, S_2\} \cos^2 2\alpha + \{S_2, S_2\} \sin 2\alpha \cdot \cos 2\alpha \\ &= -(\{S_1, D_1\} + \{S_2, D_2\}) \sin 2\alpha + (\{S_1, S_2\} - \{D_1, D_2\}) \cos 2\alpha \\ &\quad + (\{S_2, S_2\} - \{D_1, D_1\}) \cos 2\alpha \cdot \sin 2\alpha - \{S_1, D_2\} + \{D_1, S_2\} \cos^2 2\alpha \\ &\quad - \{S_2, D_1\} \sin^2 2\alpha.\end{aligned} \tag{14}$$

Since—according to (13a)—$\{S_2, D_1\} = \{D_1, S_2\}$, the sum of the last three terms in Eq. (14) is

$$-\{S_1, D_2\} + \{D_1, S_2\} \cos^2 2\alpha - \{S_2, D_1\} \sin^2 2\alpha = \{D_1, S_2\} - \{S_1, D_2\} - 2\{S_1, D_2\} \sin^2 2\alpha \tag{15}$$

and not $\{D_1, S_2\} - \{S_1, D_2\}$ as in Bahr (1991).

According to Eq. (15) and to the corresponding relationship between the two elements in the right column, Eqs. (5a) and (5b) with two unknowns: δ and α, have the following form:

$$-(A_1 + A_2) \sin 2\alpha + (B_1 + B_2) \cos 2\alpha + (C_1 + C_2) + E_2^m \cos 2\alpha \cdot \sin 2\alpha - F_2^m \sin^2 2\alpha = 0, \tag{16a}$$
$$-(A_1 - A_2) \sin 2\alpha + (B_1 - B_2) \cos 2\alpha - (C_1 - C_2) + E_2^m \cos 2\alpha \cdot \sin 2\alpha - F_2^m \sin^2 2\alpha = 0, \tag{16b}$$

where A_1, A_2, B_1, B_2, C_1 and C_2 are the same as in Eqs. (10a), (10b) and (10c), and E_2^m and F_2^m are as follows:

$$E_2^m = (\{S_2, S_2\} - \{D_1, D_1\}) \sin\delta, \tag{17a}$$
$$F_2^m = 2\{D_1, S_2\} \sin\delta, \tag{17b}$$

where the superscript m indicates modifications to the original equations.

In the following, all steps of the solution are the same as those of Bahr (1991), though the results are somewhat different. From the sum and difference of Eqs. (16a) and (16b), α and β can be derived from the following equations:

$$(-a_1 \sin 2\alpha + b_1 \cos 2\alpha) \cos\delta + (c_2 + e_2^m \cos 2\alpha \cdot \sin 2\alpha - f_2^m \sin^2 2\alpha) \sin\delta = 0, \quad (18a)$$

$$c_1 \cos\delta + (-a_2 \sin 2\alpha + b_2 \cos 2\alpha) \sin\delta = 0, \quad (18b)$$

where the small case letters refer to the corresponding quantities, without $\cos\delta$ and $\sin\delta$.

Finally

$$\tan(2\alpha_{1,2}) = \frac{1}{2} \frac{b_1 a_2 + a_1 b_2 + c_1 e_2^m}{a_1 a_2 - c_1 c_2 + c_1 f_2^m} \pm \sqrt{\frac{1}{4} \frac{(b_1 a_2 + a_1 b_2 + c_1 e_2^m)^2}{(a_1 a_2 - c_1 c_2 + c_1 f_2^m)^2} - \frac{b_1 b_2 - c_1 c_2}{a_1 a_2 - c_1 c_2 + c_1 f_2^m}} \quad (19)$$

Then δ can be determined from Eq. (18a) or (18b).

In the original solution (equation (30), Bahr, 1991), the terms with f_2^m are missing. Furthermore, e_2 in the original solution is not the same as the modified e_2^m.

5. Mathematical Discussion

According to Bahr (1991), this phase deviation method is valid in cases where the phase sensitive regional skew η and the regional one-dimensional indicator μ do not vanish.

Their original definitions are as follows (Bahr, 1988, 1991):

$$\eta = \frac{(|[D_1, S_2] - [S_1, D_2]|)^{\frac{1}{2}}}{|D_2|} = \frac{C^{\frac{1}{2}}}{|D_2|}, \quad (20a)$$

$$\mu = \frac{(|[D_1, S_2]| + |[S_1, D_2]|)^{\frac{1}{2}}}{|D_2|}. \quad (20b)$$

If $\eta = 0$, the $C_1 = c_1 = 0$. In Eqs. (18a) and (18b) if $c_1 = 0$, there are two mathematical cases:

1) $\sin\delta = 0$, which leads directly to

$$\tan 2\alpha = \frac{b_1}{a_1}.$$

2) $\sin\delta \neq 0$. In this case

$$\tan 2\alpha = \frac{b_2}{a_2}.$$

Since $\eta = 0$ means a perfect two-dimensional regional structure, this second case does not have any physical meaning. Consequently in Eq. (19) it is a reasonable preference to select from α_1 and α_2 the root associated with minimal δ solution, as it was suggested by Bahr (1991).

The above analysis tells that Eq. (19) does remain if $\eta = 0$. The only case when Eq. (19) does not work, is if the regional one-dimensional indicator μ is zero. With $\mu = 0$ no strike angle α is obtained and only one single impedance value can be recovered from the measured tensor.

Table 1. Comparison between the original and the corrected Bahr decomposition formulas as applied to the tensor $\begin{bmatrix} (-12.823, \ -1.077) & (48.311, 17.972) \\ (-21.499, -10.878) & (8.867, -0.378) \end{bmatrix}$.

	Original quantities	Quantitites recovered by the original formulas	Quantitites recovered by the corrected formulas
δ to be determined (deg)	−6	−3.9	−6.0
α to be determined (deg)	+30	+38.31	+30.00
The 2-D tensor	(0, 0) (34, 11) (−19.8, −13) (0, 0)	(0, 0) (50.159, 16.098) (−22.342, −13.946) (0, 0)	(0, 0) (51.772, 16.750) (−19.200, −12.606) (0, 0)
φ_{xy} in the 2-D tensor (deg)	17.928	17.794	17.928
φ_{yx} in the 2-D tensor (deg)	−146.712	−148.028	−146.712

6. Comparison with the Original Solution

Equation (19) in this paper and equation (30) in the paper by Bahr (1991) may result in quite significant differences in the regional strike estimates, as it is illustrated in Table 1.

In our numerical example an impedance tensor having reasonable μ and η values and some different distortion tensors were selected. The tensor was rotated from its principal direction with an angle of $\alpha = -30^\circ$ and the original strike direction had to be reconstructed, by using the two formulas.

As it is shown in Table 1, while Eq. (19) found the correct value ($\alpha = 30.0^\circ$ and $\delta = -6.0^\circ$), by using the original formula $\alpha = 38.3^\circ$ and $\delta = -3.9^\circ$ were obtained.

When the same tensor was rotated with $\alpha = +30^\circ$, Eq. (19) gave again the correct value ($\alpha = -30.0^\circ$ as expected), the discriminant in the original formula was negative.

It is not a role of the present paper to provide a complete numerical discussion. A comparison between the original and the corrected formulas clearly justifies the necessity of the proposed correction in Bahr's "phase-deviation" tensor-decomposition formulas.

7. Conclusion

According to numerical tests carried out on synthetic models, the differences between the two formulas cannot be neglected. Therefore, it is recommended to use this improved tensor decomposition method on field data.

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