Quasi-analytical solutions for APSIDAL motion in the three-body problem: Sun—minor planet—Jupiter

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This paper deals with the effect of a third body on the apsidal motion of two bodies. The specific case involves a third body-planet Jupiter and the apsidal line motion of a minor planet that orbits the Sun and has its apsidal line go through the major axis of an ellipse. The third body (Jupiter) which satisfies the Langrangian solution will affect the apsidal line motion and therefore affects the ascending and descending motions of the minor planet. In this case no analytical solutions can be obtained, and therefore specific assumptions are made along with numerical solutions. For convenience, we adopt the Lagrangian solution in the three-body problem and obtain quasi-analytical results, which are used to evaluate the effect of the planet on the dΩ/dt (Ω ascending node) of each minor planet. This method is beneficial for improving our knowledge of the orbital elements of the asteroids, and perhaps even much smaller effects such as the effects of the planets on the interplanetary dust complex. Information on the latter may be provided by using this method to investigate Jupiter's effect on the inclination of the symmetry surface of the zodiacal dust cloud.

1. Introduction

Cowling (1938) discussed the motion of the apsidal line in close binary systems by making the assumption that the shape of the stars at any instant closely approximates the equilibrium form. Considering this we attempt here to study **the effect of a third body on the motion of the apsidal line**. Specifically, we apply our analysis of this problem to situations in the solar system in an effort to further refine our knowledge of the gravitational effects of the planet Jupiter on other minor planets such as the asteroids. Later, it may be possible to delineate minute effects of Jupiter on the orbital elements of the symmetry surface of the zodiacal dust cloud.

The three-body problem here is assumed to be the Sun, an orbiting dust particle with its distance from the Sun much greater than the Sun's diameter, and the third body being Jupiter. Kopal (1959) showed that the effect of the third body will be very complicated and that analytical solutions cannot be obtained.

In dealing with the effect of **a third body (that satisfies the Lagrangian solution)** on the three-body problem, we make some specific assumptions and use numerical calculations. But before going into the mathematical details we give here a short summary of the importance of this work in evaluating the motion of the apsidal line (which is the major axis of the ellipse of the minor planet to the Sun) and searching for minute gravitational perturbations of Jupiter on the ascending node of the "Symmetry Surface" of the zodiacal cloud.

2. The Asteroids

Recently there has been a great deal of interest in gathering more detailed information on the orbital elements of the asteroids in view of their possible catastrophic collision with our planet. For this reason, we believe that improvements in determining the effects of Jupiter on the asteroid's orbital elements are worth pursuing.

3. The "Symmetry Surface" of the Zodiacal Cloud

The zodiacal light (hereafter **ZL**) arises from sunlight scattered by small (mainly 10–100 μm) dust particles which are present in interplanetary space. It is appropriate here to define the so-called "Symmetry Plane" of the ZL, which we prefer to call **"Symmetry Surface"** (Misconi *et al.*, 1990). The "Symmetry Plane" is classically defined as the plane that contains the highest number density of interplanetary dust particles and therefore the maximum brightness of the ZL. The word symmetry comes from the observation that the dust density and therefore the brightness intensity falls off in a similar fashion above and below the plane. This is also the same as searching for the "photometric axis" (locus of points of maximum brightness) of the zodiacal light.

Based on observational evidence, Misconi (1977) and Misconi and Weinberg (1978) suggested that there is no symmetry plane per se, but, rather a **"multiplicity" of planes**. That is why we prefer now to call this "multiplicity" of planes the "symmetry surface". This follows from the observation that the orientation of any symmetry plane is not constant with heliocentric distance and appears to follow closely the orbital planes of Venus or, Mars or and Jupiter, at their respective distances (Misconi, 1980; Gustafson and Misconi, 1986; Gustafson *et al.*, 1987a,b). Misconi (1977) suggested

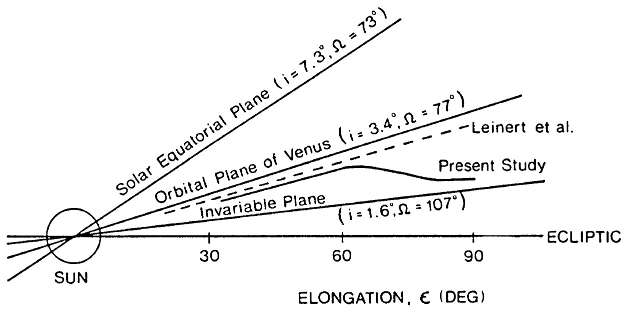

Fig. 1. A sketch of the relative inclinations from the ecliptic plane as a function of elongation: for the solar equatorial plane, the orbital plane of Venus and the invariable plane. Also shown is the position of the symmetry plane found by Leinert *et al.* (1980, dashed line) and our combined results over this range of elongation (Misconi, 1980).

further that the zodiacal dust is influenced gravitationally by the planets and that this could explain the warping of the plane.

Several recent publications were brought to the attention of the authors that touch on the subject of the "symmetry surface" of the zodiacal light: Ishiguro *et al.* (1998), James *et al.* (1997), and Ishiguro *et al.* (1997). These publications injected renewed interest in this subject but they do not affect the background or other aspects of this paper.

4. The Equations of Motion of the Three-Body Problem

Considering these realities, we start with the equations of motion of the three-body problem:

$$\ddot{\mathbf{r}}_1 = -Gm_2\frac{\mathbf{r}_1 - \mathbf{r}_2}{r_{12}^3} - Gm_3\frac{\mathbf{r}_1 - \mathbf{r}_3}{r_{13}^3}, \quad (1)$$

$$\ddot{\mathbf{r}}_2 = -Gm_3\frac{\mathbf{r}_2 - \mathbf{r}_3}{r_{23}^3} - Gm_1\frac{\mathbf{r}_2 - \mathbf{r}_1}{r_{21}^3}, \quad (2)$$

$$\ddot{\mathbf{r}}_3 = -Gm_1\frac{\mathbf{r}_3 - \mathbf{r}_1}{r_{31}^3} - Gm_2\frac{\mathbf{r}_3 - \mathbf{r}_2}{r_{32}^3}. \quad (3)$$

Using these equations, $(1) * m_1 + (2) * m_2 + (3) * m_3$ where m_1 is the mass of the Sun, m_2 is the mass of the minor planet, and m_3 is the mass of the perturbing planet (Jupiter), which then gets reduced to,

$$m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 + m_3\ddot{\mathbf{r}}_3 = 0.$$

By integral, it yields:

$$m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 + m_3\dot{\mathbf{r}}_3 = \text{constant}.$$

By selecting the center of mass as the new origin, we get:

$$\mathbf{r}_1' = \mathbf{r}_1 - \mathbf{00}', \quad \mathbf{r}_2' = \mathbf{r}_2 - \mathbf{00}', \quad \mathbf{r}_3' = \mathbf{r}_3 - \mathbf{00}'.$$

So,

$$m_1\dot{\mathbf{r}}_1' + m_2\dot{\mathbf{r}}_2' + m_3\dot{\mathbf{r}}_3' = (m_1 + m_2 + m_3)\dot{\mathbf{0}}\mathbf{0}' = M\dot{\mathbf{R}}$$

where $\dot{\mathbf{R}}$ is a constant for the center of mass. Now with respect to the new origin, we have:

$$m_1\dot{\mathbf{r}}_1'' + m_2\dot{\mathbf{r}}_2'' + m_3\dot{\mathbf{r}}_3'' = \mathbf{0}.$$

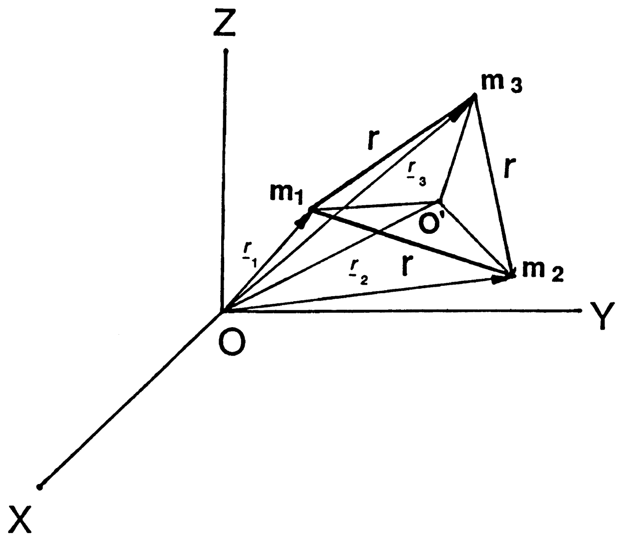

Fig. 2. Shows the masses of the Sun (m_1), the minor planet (m_2), and the planet Jupiter (m_3); $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$.

By integral, we have

$$m_1\mathbf{r}_1'' + m_2\mathbf{r}_2'' + m_3\mathbf{r}_3'' + M\dot{\mathbf{R}}' = \mathbf{0}. \quad (4)$$

In order to obtain an equilibrium solution in this system, we assume that:

$$r_{23} = r_{31} = r_{12} = r(t).$$

Then substitute them into Eq. (1):

$$\ddot{\mathbf{r}}_1 = -\frac{G}{r^3}[m_2(\mathbf{r}_1 - \mathbf{r}_2) + m_3(\mathbf{r}_1 - \mathbf{r}_3)].$$

From

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3 = \mathbf{0},$$

there exists

$$m_2(\mathbf{r}_1 - \mathbf{r}_2) + m_3(\mathbf{r}_1 - \mathbf{r}_3) = \mathbf{r}_1(m_1 + m_2 + m_3). \quad (5)$$

Taking the square of both sides of Eq. (5),

$$\begin{aligned}\mathbf{r}_1^2&(m_1 + m_2 + m_3)^2\\ &= m_2r_{12}^2 + 2m_2m_3r_{12}r_{13}\,\mathbf{cos}\,60^\circ + m_3^2r_{13}^2\\ &= r^2(m_2^2 + m_2m_3 + m_3^2).\end{aligned}$$

So,

$$\ddot{\mathbf{r}}_1 = -GM_1\frac{\mathbf{r}_1}{r_1^3},$$

where

$$M_1 = \frac{(m_2^2 + m_2m_3 + m_3^2)^{\frac{3}{2}}}{(m_1 + m_2 + m_3)^2}.$$

Hence m_2 moves in a central orbit around the center of mass, as though the mass M_1 was located there. Now if the configuration of the three bodies is maintained, then similar results will follow for the other two bodies.

As long as the initial conditions are right, the figure remains as an equilateral triangle; this condition means that $F_1 : F_2 : F_3 = r_1 : r_{2:} : r_3$, where F_i is the force per unit mass (Danby, 1962), and the resultant force acting on m_i which passes through the center of mass. Thus,

$$\frac{r_{23}}{r^\circ{}_{23}} = \frac{r_{31}}{r^\circ{}_{31}} = \frac{r_{12}}{r^\circ{}_{12}} = \lambda(t),$$

where the zero superscript indicates the value at t_0, and

$$\dot\theta_1 = \dot\theta_2 = \dot\theta_3 = \dot\theta(t),$$

i.e., the orbital angular velocities of the three bodies are the same, though they would vary with time. The total angular momentum of the system about the new origin is

$$m_1 r_1^2 \dot f_1 + m_2 r_2^2 \dot f_2 + m_3 r_3^2 \dot f_3 = (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2)\lambda^2 \dot\theta(t) = \text{constant}.$$

The angular momentum for each individual mass about the system is constant too.

Now we evaluate the relationship among r_1, r_2, r_3 and r. From

$$\ddot{\mathbf{r}}_1 = -GM_1 \frac{\mathbf{r}_1}{r_1^3}$$

we can get

$$\mathbf{F}_1 = -GM_1 \frac{\mathbf{r}_1}{r_1^3}.$$

Since $F_1 : F_2 : F_3 = r_1 : r_2 : r_3$, we might assume

$$\frac{F_1}{r_1} = \frac{F_2}{r_2} = \frac{F_3}{r_3} = S.$$

So,

$$r_1 S = GM_1 \frac{r_1}{r_1^3} \quad \text{or} \quad r_1^3 = \frac{GM_1}{S}.$$

So,

$$\begin{aligned} r_1 &= \sqrt[3]{\frac{GM_1}{S}} \\ &= \sqrt[3]{\frac{G}{S(m_1+m_2+m_3)^2}}(m_2^2 + m_2 m_3 + m_3^2)^{1/2} \\ &= S'(m_1^2 + m_2 m_3 + m_3^2)^{1/2}, \quad \text{where} \\ S' &= \sqrt[3]{\frac{G}{S(m_1+m_2+m_3)^2}}. \end{aligned}$$

Same as above for r_2 and r_3.

The next step is to calculate the distance, r, among the three bodies.

$$\begin{aligned} &r_1^2 + r_2^2 - 2r_1 r_2 \cos A = r_2 \quad \text{i.e.,} \quad \cos A = \frac{r_1^2+r_2^2-r^2}{2r_1r_2}. \\ &r_2^2 + r_3^2 - 2r_2 r_3 \cos B = r_2 \quad \text{i.e.,} \quad \cos B = \frac{r_2^2+r_3^2-r^2}{2r_2r_3}. \\ &r_3^2 + r_1^2 - 2r_3 r_1 \cos C = r_2 \quad \text{i.e.,} \quad \cos C = \frac{r_3^2+r_1^2-r^2}{2r_3r_1}. \\ &A + B + C = 360^\circ. \end{aligned}$$

Angles B and C will be less than 180°, so $\sin B > 0$ and $\sin C > 0$.

$$\begin{aligned} \sin B &= \sqrt{1-\cos^2 B} = \sqrt{1 - \left(\frac{r_2^2+r_3^2-r^2}{2r_2r_3}\right)^2} \\ &= \sqrt{\frac{-r^4 + 2(r_2^2+r_3^2)r^2 - (r_2^2-r_3^2)^2}{4r_2^2r_3^2}}. \\ \sin C &= \sqrt{1-\cos^2 C} = \sqrt{1 - \left(\frac{r_3^2+r_1^2-r^2)}{2r_3r_1}\right)^2} \\ &= \sqrt{\frac{-r^4 + 2(r_1^2+r_3^2)r^2 - (r_1^2-r_3^2)^2}{4r_1^2r_3^2}}. \\ \cos A &= \cos(360^\circ - (B+C)) = \cos(B+C) \\ &= \cos B \cos C - \sin B \sin C. \end{aligned}$$

Hence, (Eq. (A.1), see Appendix A)

Now moving some terms, squaring both sides and merging similar terms:

$$\begin{aligned} &(4r_3^2)r^6 - 4r_3^2(r_1^2+r_2^2+r_3^2)r^4 \\ &\quad + 4r_3^2[(r_1^4+r_2^4+r_3^4) - (r_1^2r_2^2 + r_2^2r_3^2 + r_3^2r_3^2)]r^2 = 0. \end{aligned}$$

In order to get a non-zero value of r, there exists

$$\begin{aligned} &r^4 - (r_1^2+r_2^2+r_3^2)r^2 \\ &\quad + [(r_1^4+r_2^4+r_3^4) - (r_1^2r_2^2 + r_2^2r_3^2 + r_3^2r_3^2)] = 0. \end{aligned}$$

So (Eq. (A.2), see Appendix A).

Now if we substitute the values of r_1, r_2, and r_3, i.e., $r_1 = S'(m_2^2 + m_2 m_3 + m_3^2)^{1/2} \ldots$ etc., the term containing the square root becomes:

$$\begin{aligned} &2(r_1^2r_2^2 + r_2^2r_3^2 + r_3^2r_3^2) - (r_1^4+r_2^4+r_3^4) \\ &= (r_1^2+r_2^2+r_3^2)^2 - 2(r_1^4+r_2^4+r_3^4), \\ &= [2(m_1^2+m_2^2+m_3^2) + (m_1m_2 + m_2m_3 + m_3m_1)]^2 \\ &\quad -2[(m_2^2 + m_2m_3 + m_3^2)^2 + (m_1^2 + m_1m_3 + m_3^2)^2 \\ &\quad + (m_1^2 + m_1m_2 + m_2^2)^2], \\ &= [\sqrt{3}(m_1m_2 + m_2m_3 + m_3m_1)]^2. \end{aligned}$$

So (Eq. (A.3), see Appendix A)

$r^{(2)}$ is the solution out of the triangle, so we pick up $r = r^{(1)} = (m_1 + m_2 + m_3)S'$, and now we can discuss the motion of each body.

The body m_2 is moving around the center of mass with an elliptical orbit due to the resultant centripetal force F_2. Now let us assume that the equation of motion of the body m_2 follows from:

$$r_1 = \frac{a_1(1-e_1^2)}{1+e_1 \mathbf{cos}\, f_1}, \text{ and the same case for } m_2,$$

$$r_2 = \frac{a_2(1-e_2^2)}{1+e_2 \mathbf{cos}\, f_2}.$$

But at any time

$$r_1 : r_2 : r = S'(m_2^2 + m_2m_3 + m_3^2)^{1/2} : S'(m_1^2 + m_1m_3 + m_3^2)^{1/2} : S'(m_1 + m_2 + m_3)$$

i.e.,

$$\begin{aligned} &\frac{a_1(1-e_1^2)}{1+e_1\cos f} : \frac{a_2(1-e_2^2)}{1+e_2\cos f} : \frac{a(1-e^2)}{1+e\cos f} \\ &= (m_2^2 + m_2m_3 + m_3^2)^{1/2} : (m_1^2 + m_1m_3 + m_3^2)^{1/2} : \\ &\quad (m_1 + m_2 + m_3). \end{aligned}$$

This means that the linear terms have to be proportional to each other, i.e.,

$$a_1 = a_2 = a$$
$$(m_2^2 + m_2m_3 + m_3^2)^{1/2} : (m_1^2 + m_1m_3 + m_3^2)^{1/2} : (m_1 + m_2 + m_3)$$

and $e_1 = e_2 = e$ and $f_1 = f_2 = f$. From

$$\frac{a}{a_1} = \frac{(m_1 + m_2 + m_3)}{(m_2^2 + m_2m_3 + m_3^2)^{1/2}} \quad \text{and} \quad \frac{a}{a_2} = \frac{(m_1 + m_2 + m_3)}{(m_1^2 + m_1m_3 + m_3^2)^{1/2}},$$

we can get

$$a = a_1\frac{(m_1 + m_2 + m_3)}{(m_2^2 + m_2m_3 + m_3^2)^{1/2}} \quad \text{and} \quad a = a_2\frac{(m_1 + m_2 + m_3)}{(m_1^2 + m_1m_3 + m_3^2)^{1/2}}.$$

Considering our problem, the vector r will move around the center of mass of m_1 and m_2, and in our case the center of mass is actually the center of the Sun. The vector r moves around the center of the mass in an ellipse whose anomaly and eccentricity follow from r_1 and r_2. Consequently, we can solve the motion of the apsidal line by considering the perturbation effect of the third body m_3, which is a planet.

We now start by considering the three masses m_1, m_2, and m_3 which are incompressible bodies with densities ρ_1, ρ_2 and ρ_3 and with their centers of mass O_1, O_2, and O_3. m_2, and m_3 rotate about the axis O_2Z_2, and O_3Z_3, respectively, and perpendicular to the plane of the triangle of their orbit. Their angular velocities $\dot\theta_1$, $\dot\theta_2$, and $\dot\theta_3$ are about these axes, with their rigid masses m_1, m_2, and m_3. Co-ordinate axes $O_1X_1Y_1Z_1$, $O_2X_2Y_2Z_2$, and $O_3X_3Y_3Z_3$ are taken with the O_1Z_1, O_2Z_2, and O_3Z_3 as axes with rotating angular velocities, and, about these axes.

The distance r between the Sun (m_1), and the minor planet (m_2), is assumed to be much larger than the diameter of the Sun. This means that the Sun and the minor planet can be regarded as forming a high precision ellipsoid. The distortion in these three assumed spheres (Sun, minor planet, planet), of radii R_1, R_2, and R_3, as the element of dm_1 of m_1 moves from x_1, y_1 and z_1 to $x_1 + \xi_1$, $y_1 + \eta_1$, $z_1 + \zeta_1$, where

$$\begin{bmatrix} \xi_1 \\ \eta_1 \\ \zeta_1 \end{bmatrix} = \begin{bmatrix} a_1 & d_1 & 0 \\ e_1 & b_1 & 0 \\ 0 & 0 & c_1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \quad \text{and}$$

$$\begin{bmatrix} a_1 & d_1 & 0 \\ e_1 & b_1 & 0 \\ 0 & 0 & c_1 \end{bmatrix} \text{ is strain matrix.}$$

The velocity of dm_1 relative to O_1 is:

$$\mathbf{V}_1 = \begin{bmatrix} V_{1x} \\ V_{1y} \\ V_{1z} \end{bmatrix} = \mathbf{V}_{01} + \dot\theta_1 x \mathbf{r}_1$$
$$= \frac{d}{dt}\begin{bmatrix} x_1 + \xi_1 \\ y_1 + \eta_1 \\ z_1 + \zeta_1 \end{bmatrix} + \begin{bmatrix} i & j & k \\ 0 & 0 & \dot\theta_1 \\ x_1 + \zeta_1 & y_1 + \eta_1 & z_1 + \zeta_1 \end{bmatrix}$$
$$= \begin{bmatrix} \dot a_1 x_1 + \dot d_1 y_1 - (y_1 + \eta_1)\dot\theta_1 \\ \dot e_1 x_1 + \dot b_1 y_1 + (x_1 + \xi_1)\dot\theta_1 \\ \dot c_1 z_1 \end{bmatrix}.$$

For m_2 and m_3, which is the same as for m_1. The total moment of the momentum of m_1 about O_1Z_1 is:

$$\sum (x_1' V_{1y} - y_1' V_{1x}) dm_1$$
$$= \sum dm_1 (x_1 + \xi_1)[\dot e_1 x_1 + \dot b_1 y_1 + (x_1 + \xi_1)\dot\theta_1]$$
$$-(y_1 + \eta_1)[\dot a_1 x_1 + \dot d_1 y_1 - (y_1 + \eta_1)\dot\theta_1]$$
$$= \dot\theta_1 \sum (r_1')^2 dm_1$$
$$= \dot\theta_1 \sum [(x_1 + \xi_1)^2 + (y_1 + \eta_1)^2] dm_1,$$

and

$$\sum x_1^2 dm_1 = \sum y_1^2 dm_1 = \frac{1}{5} m_1 R_1^2,$$
$$\sum x_1 y_1 dm_1 = 0.$$

So

$$\dot e_1(1 + a_1) + \dot b_1 d_1 - \dot a_1 e_1 - \dot d_1(1 + b_1) = 0.$$

a_1, b_1, c_1, d_1, and e_1 are very small,
so we can regard $\dot e_1 = \dot d_1$, $e_1 = d_1$. (6)

In the view of the incompressible body, there is:

$$dx_1 dy_1 dz_1 = dx_1' dy_1' dz_1' = \begin{bmatrix} 1 + a_1 & d_1 & 0 \\ e_1 & 1 + b_1 & 0 \\ 0 & 0 & 1 + c_1 \end{bmatrix} dx_1 dy_1 dz_1$$

and by neglecting the products of small terms, we have $a_1 + b_1 + c_1 = 0$. (7)

The originally spherical surface of m_1 has undergone a radial displacement, and the gravitational potential of m_1 is:

$$U_s' = -\int_0^{R_1} \frac{G dm_1}{r'} = \frac{1}{2} G m_1 \frac{(r_1^2 - 3R_1^2)}{R_1^3}.$$

The additional potential due to the distortion U' is:

$$-\frac{3}{5} G m_1 \frac{(a_1 x_1^2 + b_1 y_1^2 + c_1 z_1^2 + 2d_1 x_1 y_1)}{R_1^3}$$

The total potential is (Cowling, 1938):

$$U = \frac{1}{2} G m_1 \frac{(r_1^2 - 3R_1^2)}{R_1^3} - \frac{3}{5} G m_1 \frac{(a_1 x_1^2 + b_1 y_1^2 + c_1 z_1^2 + 2d_1 x_1 y_1)}{R_1^3}$$

The potential energy of m_1 due to its own gravitational attraction is:

(1) sphere-symmetrical part:
Force function $F_r = -\frac{\partial U_s}{\partial r} = -\frac{G m_1 r_1}{R_1^3}$,
Potential energy

$$Vs = \int_0^{R_1} dm F_r (r_1 - 0) = \int_0^{R_1} 4\pi r_1^2 dr_1 \left(-\frac{G m_1 r_1}{R_1^3}\right) r_1 = -\frac{3}{5}\frac{G m_1^2}{R_1}.$$

(2) distortional part:
From strain potential (Eq. (A.4), see Appendix A)

Total potential energy

$$V_1 = V_s + V_d = -\frac{Gm_1^2}{R_1}\left[\frac{3}{5} - \frac{2}{25}(a_1^2 + b_1^2 + c_1^2 + 2d_1^2)\right].$$

From $a_1 + b_1 + c_1 = 0$, we have

$$\begin{aligned} a_1^2 + b_1^2 + c_1^2 + 2c_1(a_1 + b_1) + 2a_1b_1 \\ = a_1^2 + b_1^2 - c_1^2 + 2a_1b_1 = 0. \end{aligned}$$

So $V_1 = -\frac{Gm_1^2}{R_1}[\frac{3}{5} - \frac{1}{25}(3c_1^2 + f_1^2 + 4d_1^2)]$, where $f_1 = a_1 - b_1$. Same cases for m_2 and m_3.

The potential energy due to the mutual attraction of m_1 and m_2 can be found from Appendix I.

$$\begin{aligned} V_{12} = -\frac{Gm_1m_2}{r} \\ \cdot\left[1 + \frac{3}{10} * \frac{R_1^2}{r^2}(-c_1 + f_1\cos 2I + 2d_1 \sin 2I)\right. \\ \left. + \frac{3}{10} * \frac{R_2^2}{r_2}(-c_2 + f_2 \cos 2II + 2d_2 \sin 2II)\right]. \end{aligned}$$

Assume O_1O, O_2O, and O_3O are the directions at equilibria for O_1X_1, O_2X_2, and O_3X_3; for the angle between the axes of O_1X_1 and OX, we have

$$I = \theta - \theta_{12}^{(1)} = \phi_1.$$

$$\cos I = \frac{r_1^2 + r^2 - r_2^2}{2r_1r} = \frac{2m_2 + m_3}{2(m_2^2 + m_2m_3 + m_3^2)^{1/2}},$$

$$\sin I = \frac{\sqrt{3}m_3}{2(m_2^2 + m_2m_3 + m_3^2)^{1/2}},$$

$$\sin 2I = \frac{\sqrt{3}m_3(2m_2 + m_3)}{2(m_2^2 + m_2m_3 + m_3^2)},$$

$$\cos 2I = \frac{(2m_2^2 + 2m_2m_3 - m_3^2)}{2(m_2^2 + m_2m_3 + m_3^2)},$$

In the case of mass m_2, $II = \theta - \theta_{12}^{(2)} = \phi_2 + 120°$.

$$\cos\phi_2 = \frac{r_2^2 + r^2 - r_3^2}{2r_2r} = \frac{2m_3 + m_1}{2(m_1^2 + m_1m_3 + m_3^2)^{1/2}},$$

$$\sin\phi_2 = \frac{\sqrt{3}m_1}{2(m_1^2 + m_1m_3 + m_3^2)^{1/2}},$$

$$\cos II = \cos(\phi_2 + 120°) = -\frac{2m_1 + m_3}{2(m_1^2 + m_1m_3 + m_3^2)^{1/2}},$$

$$\sin II = \frac{\sqrt{3}m_3}{2(m_1^2 + m_1m_3 + m_3^2)^{1/2}},$$

$$\sin 2II = \frac{\sqrt{3}m_3(2m_1 + m_3)}{2(m_1^2 + m_1m_3 + m_3^2)},$$

$$\cos 2II = \frac{(2m_1^2 + 2m_1m_3 - m_3^2)}{2(m_1^2 + m_1m_3 + m_3^2)},$$

$$II' = \theta - \theta_{23}^{(2)} = \phi_2.$$

$$\cos II' = \frac{2m_3 + m_1}{2(m_1^2 + m_1m_3 + m_3^2)^{1/2}},$$

$$\sin\phi_2 = \frac{\sqrt{3}m_1}{2(m_1^2 + m_1m_3 + m_3^2)^{1/2}},$$

$$\sin 2II' = \frac{\sqrt{3}m_1(2m_2 + m_1)}{2(m_1^2 + m_1m_3 + m_3^2)},$$

$$\cos 2II' = \frac{(2m_3^2 + 2m_3m_1 - m_1^2)}{2(m_1^2 + m_1m_3 + m_3^2)},$$

$$III' = \theta - \theta_{23}^{(2)} = \phi_3 + 120°.$$

$$\cos\phi_3 = \frac{r_3^2 + r^2 - r_1^2}{2r_3r} = \frac{2m_1 + m_2}{2(m_1^2 + m_1m_2 + m_2^2)^{1/2}},$$

$$\sin\phi_3 = \frac{\sqrt{3}m_2}{2(m_1^2 + m_1m_2 + m_2^2)^{1/2}},$$

$$\cos III' = \cos(\phi_3 + 120°) = -\frac{2m_2 + m_1}{2(m_1^2 + m_1m_2 + m_2^2)^{1/2}},$$

$$\sin III' = \frac{\sqrt{3}m_1}{2(m_1^2 + m_1m_2 + m_2^2)^{1/2}},$$

$$\sin 2III' = -\frac{\sqrt{3}m_1(2m_2 + m_1)}{(2m_1^2 + m_1m_2 + m_2^2)},$$

$$\cos 2III' = \frac{(2m_2^2 + 2m_2m_1 - m_1^2)}{2(m_1^2 + m_1m_2 + m_2^2)},$$

$$III'' = \theta - \theta_{31}^{(3)} = \phi_3.$$

$$\sin 2III'' = \frac{\sqrt{3}m_2(2m_1 + m_2)}{2(m_1^2 + m_1m_2 + m_2^2)},$$

$$\cos 2III'' = \frac{(2m_1^2 + 2m_1m_2 - m_2^2)}{2(m_1^2 + m_1m_2 + m_2^2)},$$

$$I'' = \theta - \theta_{31}^{(1)} = \phi_1 + 120°.$$

$$\cos I'' = \cos(\phi_1 + 120°) = -\frac{2m_3 + m_1}{2(m_2^2 + m_2m_3 + m_3^2)^{1/2}},$$

$$\sin I'' = \frac{\sqrt{3}m_2}{2(m_2^2 + m_2m_3 + m_3^2)^{1/2}},$$

$$\sin 2I'' = -\frac{\sqrt{3}m_2(2m_3 + m_2)}{2(m_2^2 + m_2m_3 + m_3^2)},$$

$$\cos 2I'' = \frac{2(m_3^2 + 2m_3m_2 - m_2^2)}{2(m_2^2 + m_2m_3 + m_3^2)}.$$

The kinetic energy for the system is the sum of the kinetic energy of m_1, m_2, and m_3, plus the kinetic energy of $m_1 + m_2 + m_3$, relative to their common center of mass.

The kinetic energy of m_1, m_2, and m_3, is:

$$\begin{aligned} T_1 &= \frac{1}{2}\sum(dm_1\mathbf{V}_1^2) \\ &= \frac{1}{2}\sum\{[\dot{a}_1x_1 + \dot{d}_1y_1 - \dot{\theta}_1(y_1 + \eta_1)]^2 \\ &\quad + [\dot{b}_1y_1 + \dot{d}_1x_1 + \dot{\theta}_1(x_1 + \xi_1)]^2 + c_1z_1^2\}dm_1 \\ &= \frac{1}{20}m_1R_1^2\{(\dot{f}_1 - 2d_1\dot{\theta}_1)^2 + (2\dot{d}_1 + f_1\dot{\theta}_1)^2 \\ &\quad + 3\dot{c}_1^2 + \dot{\theta}_1^2(2 - c_1)^2\}, \end{aligned}$$

$$T_2 = \frac{1}{20}m_2R_2^2\{(\dot{f}_2 - 2d_2\dot{\theta}_2)^2 + (2\dot{d}_2 + f_2\dot{\theta}_2)^2$$

$$+3\dot{c}_2^2 + \dot{\theta}_2^2(2 - c_2)^2\},$$
$$T_3 = \frac{1}{20} m_3 R_3^2 \{(\dot{f}_3 - 2d_3\dot{\theta}_3)^2 + (2\dot{d}_3 + f_3\dot{\theta}_3)^2 + 3\dot{c}_3^2 + \dot{\theta}_3^2(2 - c_3)^2\}.$$

The kinetic energy relative to their center of mass, ($\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \dot{\theta}(t)$), can be evaluated from:

$$r_1 = \frac{e_1(1 - e_1^2)}{1 + e_1 \cos f_1} \quad \text{and}$$
$$\dot{r}_1 = \frac{a_1(1 - e_1)(e_1 \mathbf{sin}\, f_1)}{(1 + e_1 \mathbf{cos}\, f_1)^2}\dot{\theta}_1 = \frac{r_1 e \,\mathbf{sin}\, f}{1 + e \,\mathbf{cos}\, f}\dot{\theta},$$

we have

$$r = \frac{a(1 - e^2)}{1 + e^* \cos f} \quad \text{and} \quad \dot{r} = \frac{r^* e^* \sin f}{(1 + e^* \cos f)}\dot{\theta}.$$

So,

$$r_1 = \frac{(a_1 r)}{a} \quad \text{and} \quad \dot{r}_1 = \frac{(a_1 \dot{r})}{a}.$$
$$T_{10} = \frac{1}{2} m_1(\dot{r}^2 + r^2\dot{\theta}^2) * \left(\frac{a_1}{a}\right)^2,$$
$$T_{20} = \frac{1}{2} m_2(\dot{r}^2 + r^2\dot{\theta}^2) * \left(\frac{a_2}{a}\right)^2,$$
$$T_{30} = \frac{1}{2} m_3(\dot{r}^2 + r^2\dot{\theta}^2) * \left(\frac{a_3}{a}\right)^2.$$

By using the expressions which we had earlier for

$$T_{10} + T_{20} + T_{30} = \frac{1}{2}\frac{(\dot{r}^2 + r^2\dot{\theta}^2)(m_1 a_1^2 + m_2 a_2^2 + m_3 a_3^2)}{a^2} = \frac{1}{2} M(r^2 + r^2\dot{\theta}^2)$$

where (Eq. (A.5), see Appendix A)

The Lagrangian function is:

$$L = T - V = T_1 + T_2 + T_3 + T_0 + T_{10} + T_{20} + T_{30} - V_1 - V_2 - V_3 - V_{12} - V_{23} - V_{31} \quad (8)$$

where

$$T_o = \frac{1}{2}(m_1 + m_2 + m_3)V^2.$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{c}_1}\right) - \frac{\partial L}{\partial c_1} = 0,$$
$$\ddot{c}_1 + \frac{1}{3}(2 - c_1)\dot{\theta}_1^2 = -\frac{4}{5} G m_1 \frac{c_1}{R_1^3} - G\frac{(m_2 + m_3)}{r^3}. \quad (9)$$

For the general co-ordinate system f_1, there exists:

$$\ddot{f}_1 - 4\dot{d}_1\dot{\theta}_1 - 2d_1\ddot{\theta}_1 - f_1\dot{\theta}_1^2 = -\frac{4}{5} G m_1 \frac{f_1}{R_1^3} + 3G\frac{(2m_2^2 - m_2 m_3 + 2m_3^2)(m_2 + m_3)}{2r^3(m_2^2 + m_2 m_3 + m_3^2)}, \quad (10)$$

$$\ddot{d}_1 + \dot{f}_1\dot{\theta}_1 + \frac{1}{2} f_1\ddot{\theta}_1 - d_1\dot{\theta}_1^2 = -\frac{4}{5}\frac{G m_1 d_1}{R_1^3} + \frac{3\sqrt{3}(G m_2 m_3(m_2 - m_3)}{4r^3(m_2^2 + m_2 m_3 + m_3^2)}. \quad (11)$$

For the general co-ordinate system θ_1 where: (Eq. (A.6), see Appendix A)

Neglecting products like $f_1\dot{d}_1$, $\dot{f}_1 d_1$, $f_1^2\dot{\theta}_1$, $d_1^2\dot{\theta}1, \ldots$ etc., we can simplify it as: (Eq. (A.7), see Appendix A)

For the general co-ordinate $\dot{r}$, we have:

$$\ddot{r} - r\dot{\theta}_1^2 = -\frac{G m_1 m_2}{M r^2} \cdot \left[1 + \frac{9R_1^2}{10r^2}(-c_1 + f_1 \mathbf{cos}\, 2I + 2d_1 \mathbf{sin}\, 2I) + \frac{9R_2^2}{10r^2}(-c_2 + f_2 \mathbf{cos}\, 2II + 2d_2 \mathbf{sin}\, 2II)\right]$$
$$-\frac{G m_2 m_3}{M r^2} \cdot \left[1 + \frac{9R_2^2}{10r^2}(-c_2 + f_2 \mathbf{cos}\, 2II' + 2d_2 \mathbf{sin}\, 2II') + \frac{9R_3^2}{10r^2}(-c_3 + f_3 \mathbf{cos}\, 2III' + 2d_3 \mathbf{sin}\, 2III')\right]$$
$$-\frac{G m_3 m_1}{M r^2}\left[1 + \frac{9R_3^2}{10r^2}(-c_3 + f_3 \mathbf{cos}\, 2III'' + 2d_3 \mathbf{sin}\, 2III'') + \frac{9R_1^2}{10r^2}(-c_1 + f_1 \mathbf{cos}\, 2I'' + 2d_1 \mathbf{sin}\, 2I'')\right]. \quad (12)$$

For the general co-ordinate $\dot{\theta}$, we have:

$$\frac{d}{dt}(r^2\dot{\theta}) = \frac{3G m_1 m_2}{5M r^2}[(-f_1 \sin 2I + 2d_1 \cos 2I)R_1^2 + (-f_2 \sin 2II + 2d_2 \cos 2II)]R_2^2$$
$$+\frac{3G m_2 m_3}{5M r^2}[(-f_2 \sin 2II' + 2d_2 \cos 2II')R_2^2 + (-f_3 \sin 2III' + 2d_3 \cos 2III')]R_3^2$$
$$+\frac{3G m_3 m_1}{5M r^2} \cdot [(-f_3 \sin 2III'' + 2d_3 \cos 2III'')R_3^2 + (-f_1 \sin 2I'' + 2d_1 \cos 2I'')]R_1^2. \quad (13)$$

The quasi-equilibrium of m_1 under the gravitational attraction of m_2 and m_3, as well as the centrifugal force due to its own rotation can be set up by neglecting $\ddot{c}_1$, $\ddot{f}_1$, $\ddot{d}_1$, $\dot{f}_1$, $\dot{d}_1, \ldots$ etc. So from (9), (10) and (11), we can get:

$$c_1\left(\frac{4}{5}\frac{G m_1}{R_1^3} - \frac{1}{3}\dot{\theta}_1^2\right) = -\frac{G(m_2 + m_3)}{r^3} - \frac{2}{3}\dot{\theta}_1^2,$$
$$f_1\left(\frac{4}{5}\frac{G m_1}{R_1^3} - \dot{\theta}_1^2\right) = \frac{3G(2m_2^2 - m_2 m_3 + 2m_3^2)(m_2 + m_3)}{2r^3(m_2^2 + m_2 m_3 + m_3^2)},$$
$$d_1\left(\frac{4}{5}\frac{G m_1}{R_1^3} - \dot{\theta}_1^2\right) = \frac{3\sqrt{3}G m_2 m_3(m_2 + m_3)}{4r^3(m_2^2 + m_2 m_3 + m_3^2)}.$$

$\dot{\theta}_1$, $\dot{\theta}_2$ and $\dot{\theta}_3$ will be the same order as the $\dot{\theta}(t)$, but $\dot{\theta}(t)$ is small too, so

$$c_1 = -\frac{5(m_2 + m_3)}{4m_1 r^3} R_1^3 - \frac{5R_1^3}{6G m_1}\dot{\theta}_1^2, \quad (14)$$

$$f_1 = \frac{15R_1^3(2m_2^2 - m_2m_3 + 2m_3^2)(m_2+m_3)}{8m_1r^3(m_2^2+m_2m_3+m_3^2)}, \quad (15)$$

$$d_1 = \frac{15\sqrt{3}R_1^3 m_2m_3(m_2m_3)}{16m_1r^3(m_2^2+m_2m_3+m_3^2)}. \quad (16)$$

The equilibrium form can be set up in the state of relative rest, while $\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \dot{\theta}$, and assume that Jupiter and the minor planet move with forms unchanged in a circular orbit around the center of mass.

$$\begin{aligned}\dot{\theta}^2 = {} & \frac{Gm_1m_2}{Mr^2}\left[1 + \frac{9R_1^2}{10R^2}(-c_1 + f_1\cos 2I + 2d_1\sin 2I)\right. \\ & \left. + \frac{9R_2^2}{10R^2}(-c_2 + f_2\cos 2II + 2d_2\sin 2II)\right] \\ & + \frac{Gm_2m_3}{Mr^2} \\ & \cdot\left[1 + \frac{9R_2^2}{10R^2}(-c_2 + f_2\cos 2II' + 2d_2\sin 2II')\right. \\ & \left. + \frac{9R_3^2}{10R^2}(-c_3 + f_3\cos 2III' + 2d_3\sin 2III')\right] \\ & + \frac{Gm_3m_1}{Mr^2} \\ & \cdot\left[1 + \frac{9R_3^2}{10R^2}(-c_3 + f_3\cos 2III'' + 2d_3\sin 2III'')\right. \\ & \left. + \frac{9R_1^2}{10R^2}(-c_1 + f_1\cos 2I'' + 2d_1\sin 2I'')\right]. \quad (17)\end{aligned}$$

So,

$$\begin{aligned}\dot{c}_1 = {} & -\frac{5(m_2+m_3)}{4m_1r^3}R_1^3 \\ & - \frac{5R_1^3}{6Gm_1}\left[\frac{Gm_1m_2}{Mr^3} + \frac{Gm_2m_3}{Mr^3} + \frac{Gm_3m_1}{Mr^3}\right] \\ = {} & -\frac{5R_1^3}{12m_1r^3}[2m_1 + 5(m_2+m_3)], \quad (18)\end{aligned}$$

$$\dot{f}_1 = \frac{15R_1^3(2m_2^2 - m_2m_3 + 2m_3^2)(m_2-m_3)}{8m_1r^3(m_2^2+m_2m_3+m_3^2)}, \quad (19)$$

and

$$\dot{d}_1 = \frac{15\sqrt{3}R_1^3 m_2m_3(m_2-m_3)}{16m_1r^3(m_2^2+m_2m_3+m_3^2)}. \quad (20)$$

Same case for m_2 and m_3.

We now consider the small oscillations about the state of **relative rest**. So we replace $c_1, f_1, d_1, r, \theta, \theta_1, \dot{\theta}, \dot{\theta}_1, \ldots$ etc. by:

$$c_1 = \dot{c}_1 + c_1', f_1 = \dot{f}_1 + f_1', d_1 = \dot{d}_1 + d_1', r = \dot{r} + r',$$

$$\theta = \dot{\theta} + \theta', \theta_1 = \dot{\theta}_1 + \theta_1', \dot{\theta} = \ddot{\theta} + \dot{\theta}', \ldots \text{ etc.}$$

$$\begin{aligned}& \ddot{c}_1' + c_1'\left[\frac{4Gm_1}{5R_1^3} - \frac{1}{3}\dot{\theta}^2\right] + \frac{2}{3}(2-c_1)\dot{\theta}\dot{\theta}_1' \\ & = \frac{3G(m_2+m_3)}{r^4}r', \quad (21)\end{aligned}$$

$$\begin{aligned}& \ddot{f}_1' + f_1'\left[\frac{4Gm_1}{5R_1^3} - \dot{\theta}^2\right] - 4\dot{d}_1'\dot{\theta} - 2f_1\dot{\theta}\dot{\theta}_1' \\ & = -\frac{9G(2m_2^2 - m_2m_3 + 2m_3^2)(m_2+m_3)}{2r^4(m_2^2+m_2m_3+m_3^2)}r', \quad (22)\end{aligned}$$

$$\begin{aligned}& \ddot{d}_1' + d_1'\left[\frac{4Gm_1}{5R_1^3} - \dot{\theta}^2\right] + \dot{f}_1'\dot{\theta} + \frac{1}{2}f_1\ddot{\theta}_1' \\ & = -\frac{9\sqrt{3}Gm_2m_3(m_2-m_3)}{4r^4(m_2^2+m_2m_3+m_3^2)}, \quad (23)\end{aligned}$$

(Eq. (A.8), see Appendix A)

$$\begin{aligned}& \ddot{r}' - r'\dot{\theta}^2 - 2r\dot{\theta}\dot{\theta}' \\ & = \frac{2Gm_1m_2r'}{Mr^3} \\ & \cdot\left[1 + \frac{9R_1^2}{5r^2}(-c_1 + f_1\cos 2I + 2d_1\sin 2I)\right. \\ & \left. + \frac{9R_2^2}{5r^2}(-c_2 + f_2\cos 2II + 2d_2\sin 2II)\right] \\ & + \frac{2Gm_2m_3r'}{Mr^3} \\ & \cdot\left[1 + \frac{9R_2^2}{5r^2}(-c_2 + f_2\cos 2II' + 2d_2\sin 2II')\right. \\ & \left. + \frac{9R_3^2}{5r^2}(-c_3 + f_3\cos 2III' + 2d_3\sin 2III')\right] \\ & + \frac{2Gm_3m_1r'}{Mr^3} \\ & \cdot\left[1 + \frac{9R_3^2}{5r^2}(-c_3 + f_3\cos 2III'' + 2d_3\sin 2III'')\right. \\ & \left. + \frac{9R_1^2}{5r^2}(-c_1 + f_1\cos 2I'' + 2d_1\sin 2I'')\right] \\ & - \frac{9Gm_1m_2}{10Mr^4}[R_1^2(-c_1' + f_1'\cos 2I + 2d_1'\sin 2I) \\ & + R_2^2(-c_2' + f_2'\cos 2II + 2d_2'\sin 2II)] \\ & - \frac{9Gm_1m_2}{10Mr^4}[R_2^2(-c_2' + f_2'\cos 2II' + 2d_2'\sin 2II') \\ & + R_3^2(-c_3' + f_3'\cos 2III' + 2d_3'\sin 2III')] \\ & - \frac{9Gm_1m_2}{10Mr^4}[R_3^2(-c_3' + f_3'\cos 2III'' + 2d_3'\sin 2III'') \\ & + R_1^2(-c_1' + f_1'\cos 2I'' + 2d_1'\sin 2I'')], \quad (24)\end{aligned}$$

$$\begin{aligned}r\ddot{\theta}' + 2\dot{r}'\dot{\theta} = {} & \frac{3Gm_1m_2}{5Mr^4}[(-f_1'\sin 2I + 2d_1'\cos 2I)R_1^2 \\ & + (-f_2'\sin 2II + 2d_2'\cos 2II)R_2^2] \\ & + \frac{3Gm_1m_2}{5Mr^4} \\ & \cdot[(-f_2'\sin 2II' + 2d_2'\cos 2II')R_2^2 \\ & + (-f_3'\sin 2III' + 2d_3'\cos 2III')R_3^2] \\ & + \frac{3Gm_1m_2}{5Mr^4} \\ & \cdot[(-f_3'\sin 2III'' + 2d_3'\cos 2III'')R_3^2 \\ & + (-f_3'\sin 2I'' + 2d_3'\cos 2I'')R_1^2]. \quad (25)\end{aligned}$$

There are a series of solutions for those equations of small oscillations. We assume, that the period is $\frac{2\pi}{p}$, and $\dot{c}_1' = ipc_1'$, $\ddot{c}_1' = p^2c_1', \ldots$ etc.

There is one oscillation for which $p^2 \to \dot{\theta}^2$, so by substituting $\dot{r}' = ipr'$ and $\ddot{\theta}' = -p^2\theta'$ into Eq. (25), and neglecting the small terms on the RHS, we get:

$$-rp^2\theta' + 2ipr'\dot{\theta} \sim 0, \quad \theta' = \frac{2ir'\dot{\theta}}{rp}, \quad \dot{\theta}' = -\frac{2r'\dot{\theta}}{r}. \quad (26)$$

Then substituting θ' into Eq. (24):

$$-p^2 r' - \dot\theta^2 r' - 2r\dot\theta i p\left(\frac{2ir'\dot\theta}{rp}\right) \sim r'\dot\theta^2, \quad \text{where}$$

$$\dot\theta^2 \sim \frac{G}{Mr^3}(m_1m_2 + m_2m_3 + m_3m_1).$$

That is, $-p^2r' - \dot\theta^2 + 4\dot\theta^2 = \dot\theta^2$, so $p^2 \to \dot\theta^2$.

The ratio of the period of rotation of the apse to the period of orbital revolution ε (which is not the ε in Fig. 1) is:

$$\varepsilon = \frac{\frac{2\pi}{p} - \frac{2\pi}{\dot\theta}}{\frac{2\pi}{p}} = \frac{\dot\theta - p}{\dot\theta} = 1 - \frac{p}{\dot\theta} \quad \text{i.e., } p = (1-\varepsilon)\dot\theta. \quad (27)$$

From Eq. (26) we can see the θ' and $\frac{r'}{r}$ are of the same order of magnitude, while c', f', d' and θ' are of the order of $\frac{R_1 r'}{r^4}$ (from Eqs. (20), (21), (22) and (23)). So, the terms involving $\frac{4Gm_1}{5R_1^3}$ are large compared with the other terms on the left. So approximately:

$$c_1' = \frac{15(m_2+m_3)R_1^3}{4m_1r^4}r', \quad (28)$$

$$f_1' = -\frac{45(2m_2^2 - m_2m_3 + 2m_3^2)(m_2+m_3)R_1^3}{8m_1r^4(m_2^2+m_2m_3+m_3^2)}r', \quad (29)$$

$$d_1' = -\frac{45\sqrt{3}m_2m_3(m_2+m_3)R_1^3}{16m_1r^4(m_2^2+m_2m_3+m_3^2)}r'. \quad (30)$$

Same case applies for the parameters of other m_2 and m_3.

Now for Eq. (17), we first consider the terms involving R_1^2:

$$\begin{aligned}
&\frac{3Gm_1}{5Mr^3}[m_2(-f_1\sin 2I + 2d_1\cos 2I) \\
&\quad + m_3(-f_1\sin 2I'' + 2d_1\cos 2I'')] \\
&= \frac{3Gm_1}{5Mr^3}[-f_1(m_2\sin 2I + m_3\sin 2I'') \\
&\quad + 2d_1(m_2\cos 2I + m_3\cos 2I'')] \\
&= \frac{3Gm_1}{5Mr^3}\Bigg[-f_1\frac{3m_2m_3(m_2-m_3)}{2(m_2^2+m_2m_3+m_3^2)} \\
&\quad + 2d_1\frac{(2m_2^2 - m_2m_3 + 2m_3^2)(m_2+m_3)}{2(m_2^2+m_2m_3+m_3^2)}\Bigg].
\end{aligned}$$

But

$$\frac{f_1}{2d_1} = \frac{\dot f_1 + f_1'}{2(\dot d_1 + d_1')} = \frac{(2m_2^2 - m_2m_3 + 2m_3^2)(m_2+m_3)}{\sqrt{3}m_2m_3(m_2-m_3)}.$$

So, the terms involving R_1^2, R_2^2 and R_3^2 are zero. As a result,

$$\frac{d}{dt}(r^2\dot\theta) = 0. \quad (31)$$

This means that the angular momentum is invariant. Due to

$$\frac{f_1'}{\dot f_1} = \frac{d_1'}{\dot d_1} = -\frac{3r'}{r} \quad \text{and}$$

$$\frac{c_1'}{\dot c_1} = -3\left[\frac{(m_2+m_3)}{2m_1 + 5(m_2+m_3)}\right]\frac{r'}{r},$$

substituting into Eq. (24) we get:

$$\begin{aligned}
& r'\Bigg\{p^2 + \dot\theta^2 + \frac{2Gm_1m_2}{Mr^3} \\
&\cdot\Bigg[1 + \frac{9R_1^2}{5r^2}(-c_1 + f_1\cos 2I + 2d_1\sin 2I) \\
&+ \frac{9R_2^2}{5r^2}(-c_2 + f_2\cos 2II + 2d_2\sin 2II)\Bigg] \\
&+ \frac{2Gm_2m_3}{Mr^3} \\
&\cdot\Bigg[1 + \frac{9R_2^2}{5r^2}(-c_2 + f_2\cos 2II' + 2d_2\sin 2II') \\
&+ \frac{9R_3^2}{5r^2}(-c_3 + f_3\cos 2III' + 2d_3\sin 2III')\Bigg] \\
&+ \frac{2Gm_3m_1}{Mr^3} \\
&\cdot\Bigg[1 + \frac{9R_3^2}{5r^2}(-c_3 + f_3\cos 2III'' + 2d_3\sin 2III'') \\
&+ \frac{9R_1^2}{5r^2}(-c_1 + f_1\cos 2I'' + 2d_1\sin 2I'')\Bigg]\Bigg\} \\
&+ \frac{27Gm_1m_2}{10Mr^5}\Bigg\{R_1^2\Bigg[\frac{-3(m_2+m_3)c_1}{2m_1 + 5(m_2+m_3)} \\
&+ f_1\cos 2I + 2d_1\sin 2I\Bigg] \\
&+ R_2^2\Bigg[\frac{-(m_3+m_1)c_2}{2m_2 + 5(m_3+m_1)} \\
&+ f_2\cos 2II + 2d_2\sin 2II\Bigg]\Bigg\} \\
&+ \frac{27Gm_2m_3}{10Mr^5}\Bigg\{R_2^2\Bigg[\frac{-3(m_3+m_1)c_2}{2m_2 + 5(m_3+m_1)} \\
&+ f_2\cos 2II' + 2d_2\sin 2II'\Bigg] \\
&+ R_3^2\Bigg[\frac{-(m_1+m_2)c_3}{2m_3 + 5(m_1+m_2)} \\
&+ f_3\cos 2III' + 2d_3\sin 2III'\Bigg]\Bigg\} \\
&+ \frac{27Gm_3m_1}{10Mr^5}\Bigg\{R_3^2\Bigg[\frac{-3(m_1+m_2)c_3}{2m_3 + 5(m_1+m_2)} \\
&+ f_3\cos 2III'' + 2d_3\sin 2III''\Bigg] \\
&+ R_1^2\Bigg[\frac{-(m_2+m_3)c_1}{2m_1 + 5(m_2+m_3)} \\
&+ f_1\cos 2I'' + 2d_1\sin 2I''\Bigg]\Bigg\} - 4\dot\theta^2 r' = 0.
\end{aligned}$$

Substituting $\dot c_1$, $\dot f_1$ and $\dot d_1$, we have:

$$\begin{aligned}
& p^2 - 3\dot\theta^2 + \frac{2Gm_1m_2}{Mr^3}\Bigg\{\Bigg[1 + \frac{9R_1^2}{10r^2}(\ldots) + \frac{9R_2^2}{10r^2}(\ldots)\Bigg] \\
&+ \frac{9R_1^2}{10r^2}(\ldots) + \frac{9R_2^2}{10r^2}(\ldots) + \frac{27R_1^2}{20r^2}(\ldots) + \frac{27R_2^2}{20r^2}(\ldots)\Bigg\} \\
&+ \frac{2Gm_2m_3}{Mr^3}\Bigg\{\Bigg[1 + \frac{9R_2^2}{10r^2}(\ldots) + \frac{9R_3^2}{10r^2}(\ldots)\Bigg] \\
&+ \frac{9R_2^2}{10r^2}(\ldots) + \frac{9R_3^2}{10r^2}(\ldots) + \frac{27R_2^2}{20r^2}(\ldots) + \frac{27R_3^2}{20r^2}(\ldots)\Bigg\} \\
&+ \frac{2Gm_3m_1}{Mr^3}\Bigg\{\Bigg[1 + \frac{9R_3^2}{10r^2}(\ldots) + \frac{9R_1^2}{10r^2}(\ldots)\Bigg]
\end{aligned}$$

$$+\frac{9R_3^2}{10r^2}(\ldots)+\frac{9R_1^2}{10r^2}(\ldots)+\frac{27R_3^2}{20r^2}(\ldots)+\frac{27R_1^2}{20r^2}(\ldots)\Big\}$$
$$=0.$$

Substituting $\dot{\theta}^2$ from Eq. (17) into it:

$$p^2-3\dot{\theta}^2+2\dot{\theta}^2\Bigg\{1+\frac{m_1m_2}{m_1m_2+m_2m_3+m_3m_1}$$
$$\cdot\left[\frac{9R_1^2}{10r^2}(\ldots)+\frac{9R_2^2}{10r^2}(\ldots)+\frac{27R_1^2}{20r^2}(\ldots)+\frac{27R_2^2}{20r^2}(\ldots)\right]$$
$$+\frac{m_2m_3}{m_1m_2+m_2m_3+m_3m_1}$$
$$\cdot\left[\frac{9R_2^2}{10r^2}(\ldots)+\frac{9R_3^2}{10r^2}(\ldots)+\frac{27R_2^2}{20r^2}(\ldots)+\frac{27R_3^2}{20r^2}(\ldots)\right]$$
$$+\frac{m_3m_1}{m_1m_2+m_2m_3+m_3m_1}$$
$$\cdot\left[\frac{9R_3^2}{10r^2}(\ldots)+\frac{9R_1^2}{10r^2}(\ldots)+\frac{27R_3^2}{20r^2}(\ldots)+\frac{27R_1^2}{20r^2}(\ldots)\right]\Bigg\}$$
$$=0,$$
$$p^2=\dot{\theta}^2[1-2(\ldots)],$$
$$\varepsilon=1-\frac{p}{\dot{\theta}}=1-[1-2(\ldots)]^{1/2}$$
$$=\frac{9R_1^2}{10r^2(m_1m_2+m_2m_3+m_3m_1)}$$
$$\cdot\Bigg\{-(m_1m_2+m_2m_3)\left[1+\frac{9(m_2+m_3)}{2(2m_1+5(m_2+m_3))}\right]\dot{c}_1$$
$$+\frac{5}{2}(m_1m_2\cos 2I+m_3m_1\cos 2I'')\dot{f}_1$$
$$+\frac{5}{2}(m_1m_2\sin 2I+m_3m_1\sin 2I'')2\dot{d}_1\Bigg\}$$
$$+\frac{9R_2^2}{10r^2(m_1m_2+m_2m_3+m_3m_1)}$$
$$\cdot\Bigg\{-(m_2m_3+m_3m_1)\left[1+\frac{9(m_3+m_1)}{2(2m_2+5(m_3+m_1))}\right]\dot{c}_2$$
$$+\frac{5}{2}(m_2m_3\cos 2II+m_1m_2\cos 2II')\dot{f}_2$$
$$+\frac{5}{2}(m_2m_3\sin 2II+m_1m_2\sin 2II')2\dot{d}_2\Bigg\}$$
$$+\frac{9R_3^2}{10r^2(m_3m_1+m_1m_2+m_2m_3)}$$
$$\cdot\Bigg\{-(m_3m_1+m_1m_2)\left[1+\frac{9(m_2+m_3)}{2(2m_3+5(m_1+m_2))}\right]\dot{c}_3$$
$$+\frac{5}{2}(m_3m_1\cos 2II'+m_2m_3\cos 2III'')\dot{f}_3$$
$$+\frac{5}{2}(m_3m_1\sin 2III'+m_2m_3\sin 2III'')2\dot{d}_3\Bigg\}.$$

Finally, substituting $\dot{c}_1, \dot{f}_1, \dot{d}_1, \dot{c}_2, \dot{f}_2, \dot{d}_2, \dot{c}_3, \dot{f}_3, \dot{d}_3$ we have

$$\varepsilon=\frac{\left(\frac{R_1}{r}\right)^5}{m_1m_2+m_2m_3+m_3m_1}$$
$$*\Bigg\{(m_1m_2+m_3m_1)\left[\frac{3}{4}+\frac{57(m_2+m_3)}{16m_1}\right]$$
$$+\frac{135}{64}\left[\frac{(m_2+m_3)(2m_2^2-m_2m_3+2m_3^2)}{m_2^2+m_2m_3+m_3^2}\right]^2$$
$$+\frac{405}{64}\left[\frac{m_2m_3(m_2-m_3)}{m_2^2+m_2m_3+m_3^2}\right]^2\Bigg\}$$
$$+\frac{\left(\frac{R_2}{r}\right)^5}{m_1m_2+m_2m_3+m_3m_1}$$
$$*\Bigg\{(m_2m_3+m_1m_2)\left[\frac{3}{4}+\frac{57(m_3+m_1)}{16m_2}\right]$$
$$+\frac{135}{64}\left[\frac{(m_3+m_1)(2m_3^2-m_3m_1+2m_1^2)}{m_3^2+m_3m_1+m_1^2}\right]^2$$
$$+\frac{405}{64}\left[\frac{m_3m_1(m_3-m_1)}{m_3^2+m_3m_1+m_1^2}\right]^2\Bigg\}$$
$$+\frac{\left(\frac{R_3}{r}\right)^5}{m_1m_2+m_2m_3+m_3m_1}$$
$$*\Bigg\{(m_3m_1+m_2m_3)\left[\frac{3}{4}+\frac{57(m_1+m_2)}{16m_3}\right]$$
$$+\frac{135}{64}\left[\frac{(m_1+m_2)(2m_1^2-m_1m_2+2m_2^2)}{m_1^2+m_1m_2+m_2^2}\right]^2$$
$$+\frac{405}{64}\left[\frac{m_1m_2(m_1-m_2)}{m_1^2+m_1m_2+m_2^2}\right]^2\Bigg\}. \quad (32)$$

If m_3 goes to zero (and R_3 too), the result will be consistent with Cowling's result (Ziglin, 1976), i.e.:

$$\varepsilon=\frac{3}{4}\left[\left(1+16\frac{m_2}{m_1}\right)\left(\frac{R_1}{r}\right)^5+\left(1+16\frac{m_1}{m_2}\right)\left(\frac{R_2}{r}\right)^5\right].$$

We consider next the elliptical motion. To generalize Eq. (12) to apply for non-uniform bodies, the small terms involving $c_1, f_1, d_1, \ldots$ etc. can be expressed in terms of the differences between the principal moments of inertia of m_1, m_2, and m_3. For these non-uniform bodies, these values are $\frac{4k_1}{3}$, $\frac{4k_2}{3}$, and $\frac{4k_3}{3}$ times as large as for uniform bodies of the same masses and radii. Therefore,

$$r-r\dot{\theta}^2=-\frac{G(m_1m_2+m_2m_3+m_3m_1)}{Mr^2}$$
$$*\Bigg\{1+\frac{k_1R_1^5}{r^2(m_1m_2+m_2m_3+m_3m_1)}$$
$$\cdot\Bigg[\frac{(m_2+m_3)}{G}\dot{\theta}_1^2+\frac{1}{r^3}\Bigg(\frac{3(m_2+m_3)^2}{2}$$
$$+\frac{9}{8}\left(\frac{(m_2+m_3)(2m_2^2-m_2m_3+2m_3^2)}{(m_2^2+m_2m_3+m_3^2)}\right)^2$$
$$+\frac{27}{8}\left(\frac{m_2m_3(m_2-m_3)}{(m_2^2+m_2m_3+m_3^2)}\right)^2\Bigg)\Bigg]$$
$$+\frac{k_2R_2^5}{r^2(m_1m_2+m_2m_3+m_3m_1)}$$
$$\cdot\Bigg[\frac{(m_3+m_1)}{G}\dot{\theta}_2^2+\frac{1}{r^3}\Bigg(\frac{3(m_3+m_1)^2}{2}$$
$$+\frac{9}{8}\left(\frac{(m_3+m_1)(2m_3^2-m_3m_1+2m_1^2)}{(m_3^2+m_3m_1+m_1^2)}\right)^2$$
$$+\frac{27}{8}\left(\frac{m_3m_1(m_3-m_1)}{(m_3^2+m_3m_1+m_1^2)}\right)^2\Bigg)\Bigg]$$
$$+\frac{k_3R_3^5}{r^2(m_1m_2+m_2m_3+m_3m_1)}$$

$$
\cdot\left[\frac{(m_1+m_2)}{G}\dot{\theta}_3^2+\frac{1}{r^3}\left(\frac{3(m_1+m_2)^2}{2}+\frac{9}{8}\left(\frac{(m_1+m_2)(2m_3^2-m_3m_1+2m_1^2)}{(m_1^2+m_1m_2+m_2^2)}\right)^2+\frac{27}{8}\left(\frac{m_1m_2(m_1-m_2)}{(m_1^2+m_1m_2+m_2^2)}\right)^2\right)\right]\Bigg\}
$$

$$
=-\frac{G(m_1m_2+m_2m_3+m_3m_1)}{Mr^2}*[1+(\delta_1+\delta_2+\delta_3)r^{-2}+(\delta_1'+\delta_2'+\delta_3')r^{-5}] \tag{33}
$$

where

$$
\delta_1=\frac{k_1R_1^5(m_2+m_3)}{G(m_1m_2+m_2m_3+m_3m_1)}\dot{\theta}_1^2
$$

$$
\delta_2=\frac{k_2R_2^5(m_3+m_1)}{G(m_1m_2+m_2m_3+m_3m_1)}\dot{\theta}_2^2
$$

$$
\delta_3=\frac{k_3R_3^5(m_1+m_2)}{G(m_1m_2+m_2m_3+m_3m_1)}\dot{\theta}_3^2.
$$

$$
\delta_1'=\frac{k_1R_1^5}{(m_1m_2+m_2m_3+m_3m_1)}\times\left\{\frac{3(m_2+m_3)^2}{2}+\frac{9}{8}\left[\frac{(m_2+m_3)(2m_2^2-m_2m_3+2m_3^2)^2}{(m_2^2+m_2m_3+m_3^2)}\right]+\frac{27}{8}\left(\frac{m_2m_3(m_3)}{(m_2^2+m_2m_3+m_3^2)}\right)^2\right\}
$$

$$
\delta_2'=\frac{k_2R_2^5}{(m_1m_2+m_2m_3+m_3m_1)}\times\left\{\frac{3(m_3+m_1)^2}{2}+\frac{9}{8}\left[\frac{(m_3+m_1)(2m_3^2-m_3m_1+2m_1^2)^2}{(m_3^2+m_3m_1+m_1^2)}\right]+\frac{27}{8}\left(\frac{m_3m_1(m_3-m_1)}{(m_3^2+m_3m_1+m_1^2)}\right)^2\right\}
$$

$$
\delta_3'=\frac{k_3R_3^5}{(m_1m_2+m_2m_3+m_3m_1)}\times\left\{\frac{3(m_1+m_2)^2}{2}+\frac{9}{8}\left[\frac{(m_1+m_2)(2m_1^2-m_1m_2+2m_2^2)^2}{(m_1^2+m_1m_2+m_2^2)}\right]+\frac{27}{8}\left(\frac{m_1m_2(m_1-m_2)}{(m_1^2+m_1m_2+m_2^2)}\right)^2\right\}.
$$

Now from Eq. (31), we have $\frac{d}{dt}(r^2\dot{\theta})=0$; so the total angular momentum is $r^2\dot{\theta}=h$. Suppose $u=1/r$. Then from Eq. (33), we can set up:

$$
\frac{d^2u}{d\theta^2}+u=\frac{G(m_1m_2+m_2m_3+m_3m_1)}{Mh^2}(\delta u^2+\delta' u^5)
$$

where $\delta=\delta_1+\delta_2+\delta_3$ and $\delta'=\delta_1'+\delta_2'+\delta_3'$.

By using the Lagrangian method of changing constant, then the solution for a second order differential equation is:

$$
u=\left[A-\int_0^\theta\frac{G(m_1m_2+m_2m_3+m_3m_1)}{Mh^2}\cdot(\delta u^2+\delta' u^5)\sin\phi d\phi\right]\cos\theta+\left[B+\int_0^\theta\frac{G(m_1m_2+m_2m_3+m_3m_1)}{Mh^2}\cdot(\delta u^2+\delta' u^5)\cos\phi d\phi\right]\sin\theta+\frac{G(m_1m_2+m_2m_3+m_3m_1)}{Mh^2}. \tag{34}
$$

The first approximation is obtained by neglecting the terms involving δ and $\dot{\delta}$. If the initial line of θ is suitably selected ($B=0$) and the approximation then is the ellipse:

$$
lu=1=e\cos\theta \quad \text{where} \quad l=\frac{Mh^2}{G(m_1m_2+m_2m_3+m_3m_1)},\quad e=lA=\frac{AMh^2}{G(m_1m_2+m_2m_3+m_3m_1)}. \tag{35}
$$

The second approximation is obtained by substituting the result of Eq. (35) into the RHS of Eq. (34). After one revolution, the periastron longitude has increased by θ_p; and θ_p satisfies:

$$
\frac{\partial u}{\partial\theta}\Bigg|\begin{matrix}\theta=\theta_p=0\\ \theta=2\pi\end{matrix}\quad \text{is neglecting the derivitave of small terms of } (\delta u^2+\delta' u^5).
$$

We have (Eq. (A.9), see Appendix A)

Since θ_p is very small, we have approximately:

$$
\theta_p=\frac{G(m_1m_2+m_2m_3+m_3m_1)}{AMh^2}\cdot\int_0^{2\pi}(\delta u^2+\delta' u^5)\cos\phi d\phi,
$$

$$
=\frac{1}{E}\left(\delta\int_0^{2\pi}\left(\frac{1+E\cos\phi}{1}\right)\cos\phi d\phi+\delta'\int_0^{2\pi}\left(\frac{1+E\cos\phi}{1}\right)^5\cos\phi d\phi\right)
$$

$$
=\frac{1}{E}\left[\delta l^{-2}2\pi E+\delta' l^{-5}2\pi\left(\frac{5}{2}E+\frac{15}{4}E^3+\frac{5}{16}E^5\right)\right].
$$

The increment θ_p is equal to $2\pi\varepsilon$, therefore:

$$
\varepsilon=\frac{\delta}{l^2}+\frac{5}{2l^5}\delta'\left(1+\frac{3}{2}E^2+\frac{1}{8}E^4\right).
$$

Substituting $\delta=\delta_1+\delta_2+\delta_3$ and $\delta'=\delta_1'+\delta_2'+\delta_3'$

$$
\varepsilon=\frac{1}{l^2}\left(\frac{k_1R_1^5(m_2+m_3)\dot{\theta}_1^2}{G(m_1m_2+m_2m_3+m_3m_1)}+\frac{k_2R_2^5(m_3+m_1)\dot{\theta}_2^2}{G(m_1m_2+m_2m_3+m_3m_1)}\right.
$$

$$+\frac{k_3R_3^5(m_1+m_2)\dot{\theta}_3^2}{G(m_1m_2+m_2m_3+m_3m_1)}\Bigg)$$
$$+\frac{1}{l^5}\left(1+\frac{3}{2}e^2+\frac{1}{8}e^4\right)$$
$$\cdot\Bigg\{\frac{k_1R_1^5}{(m_1m_2+m_2m_3+m_3m_1)}\Bigg[\frac{15(m_2+m_3)^2}{4}$$
$$+\frac{45}{16}\left(\frac{(m_2+m_3)(2m_2^2-m_2m_3+2m_3^2)}{(m_2^2+m_2m_3+m_3^2)}\right)^2$$
$$+\frac{135}{16}\left(\frac{m_2m_3(m_2-m_3)}{(m_2^2+m_2m_3+m_3^2)}\right)^2\Bigg]$$
$$+\frac{k_2R_2^5}{(m_1m_2+m_2m_3+m_3m_1)}\Bigg[\frac{15(m_3+m_1)^2}{2}$$
$$+\frac{45}{16}\left(\frac{(m_3+m_1)(2m_3^2-m_3m_1+2m_1^2)}{(m_3^2+m_3m_1+m_1^2)}\right)^2$$
$$+\frac{135}{16}\left(\frac{m_3m_1(m_3-m_1)}{(m_3^2+m_3m_1+m_1^2)}\right)^2\Bigg]$$
$$+\frac{k_3R_3^5}{(m_1m_2+m_2m_3+m_3m_1)}\Bigg[\frac{15(m_1+m_2)^2}{4}$$
$$+\frac{45}{16}\left(\frac{(m_1+m_2)(2m_1^2-m_1m_2+2m_2^2)}{(m_1^2+m_1m_2+m_2^2)}\right)^2$$
$$+\frac{135}{16}\left(\frac{m_1m_2(m_1-m_2)}{(m_1^2+m_1m_2+m_2^2)}\right)^2\Bigg]\Bigg\}. \quad (36)$$

Rotating angular velocities with the mean orbital ones (co-rotation and co-revolution), we get:

$$\dot{\theta}_1=\dot{\theta}_2=\dot{\theta}_3=\dot{\theta} \quad \text{and} \quad \dot{\theta}_1=n_1=\sqrt{\frac{GM_1}{a_1^3}}$$

$$\dot{\theta}_1^2=\frac{G\frac{(m_2^2+m_2m_3+m_3^2)^{\frac{3}{2}}}{(m_1+m_2+m_3)^2}}{\left[a\frac{(m_2^2+m_2m_3+m_3^2)^{\frac{1}{2}}}{(m_1+m_2+m_3)}\right]^3}=G\frac{(m_1+m_2+m_3)}{a^3}=\dot{\theta}^2.$$

Also

$$a=\frac{r_a+r_p}{2}=\left(\frac{1}{1-e}+\frac{1}{1+e}\right)=\frac{1}{1-e^2}.$$

So

$$\dot{\theta}^2=\frac{G(m_1+m_2+m_3)}{l^3}(1-e^2)^3.$$

By substituting $\dot{\theta}_1$, $\dot{\theta}_2$ and $\dot{\theta}_3$ into Eq. (36), it has:

$$\varepsilon=\frac{k_1R_1^5}{l^5(m_1m_2+m_2m_3+m_3m_1)}$$
$$*\Bigg\{(1-e^2)^3(m_2+m_3)(m_1+m_2+m_3)$$
$$+\left(1+\frac{3}{2}e^2+\frac{1}{8}E^4\right)\Bigg[\frac{15(m_2+m_3)^2}{4}$$
$$+\frac{45}{16}\left(\frac{(m_2+m_3)(2m_2^2-m_2m_3+2m_3^2)}{(m_2^2+m_2m_3+m_3^2)}\right)^2$$
$$+\frac{135}{16}\left(\frac{m_2m_3(m_2-m_3)}{(m_2^2+m_2m_3+m_3^2)}\right)^2\Bigg]\Bigg\}$$
$$+\frac{k_2R_2^5}{l^5(m_1m_2+m_2m_3+m_3m_1)}$$
$$*\Bigg\{(1-e^2)^3(m_3+m_1)(m_1+m_2+m_3)$$
$$+\left(1+\frac{3}{2}e^2+\frac{1}{8}e^4\right)\Bigg[\frac{15(m_3+m_1)^2}{4}$$
$$+\frac{45}{16}\left(\frac{(m_3+m_1)(2m_3^2-m_3m_1+2m_1^2)}{(m_3^2+m_3m_1+m_1^2)}\right)^2$$
$$+\frac{135}{16}\left(\frac{m_3m_1(m_3-m_1)}{(m_3^2+m_3m_1+m_1^2)}\right)^2\Bigg]\Bigg\}$$
$$+\frac{k_3R_3^5}{l^5(m_1m_2+m_2m_3+m_3m_1)}$$
$$*\Bigg\{(1-e^2)^3(m_1+m_2)(m_1+m_2+m_3)$$
$$+\left(1+\frac{3}{2}e^2+\frac{1}{8}e^4\right)\Bigg[\frac{15(m_1+m_2)^2}{4}$$
$$+\frac{45}{16}\left(\frac{(m_1+m_2)(2m_1^2-m_1m_2+2m_2^2)}{(m_1^2+m_1m_2+m_2^2)}\right)^2$$
$$+\frac{135}{16}\left(\frac{m_1m_2(m_1-m_2)}{(m_1^2+m_1m_2+m_2^2)}\right)^2\Bigg]\Bigg\}.$$

If E is very small, we have:

$$\varepsilon=\frac{k_1}{(m_1m_2+m_2m_3+m_3m_1)}\left(\frac{R_1}{r}\right)^5$$
$$*\Bigg\{(m_2+m_3)(m_1+m_2+m_3)+\Bigg[\frac{15(m_2+m_3)^2}{4}$$
$$+\frac{45}{16}\left(\frac{(m_2+m_3)(2m_2^2-m_2m_3+2m_3^2)}{(m_2^2+m_2m_3+m_3^2)}\right)^2$$
$$+\frac{135}{16}\left(\frac{m_2m_3(m_2-m_3)}{(m_2^2+m_2m_3+m_3^2)}\right)^2\Bigg]\Bigg\}$$
$$+\frac{k_2}{(m_1m_2+m_2m_3+m_3m_1)}\left(\frac{R_2}{r}\right)^5$$
$$*\Bigg\{(m_3+m_1)(m_1+m_2+m_3)+\Bigg[\frac{15(m_3+m_1)^2}{4}$$
$$+\frac{45}{16}\left(\frac{(m_3+m_1)(2m_3^2-m_3m_1+2m_1^2)}{(m_3^2+m_3m_1+m_1^2)}\right)^2$$
$$+\frac{135}{16}\left(\frac{m_3m_1(m_3-m_1)}{(m_3^2+m_3m_1+m_1^2)}\right)^2\Bigg]\Bigg\}$$
$$+\frac{k_3}{(m_1m_2+m_2m_3+m_3m_1)}\left(\frac{R_3}{r}\right)^5$$
$$*\Bigg\{(m_1+m_2)(m_1+m_2+m_3)+\Bigg[\frac{15(m_1+m_2)^2}{4}$$
$$+\frac{45}{16}\left(\frac{(m_1+m_2)(2m_1^2-m_1m_2+2m_2^2)}{(m_1^2+m_1m_2+m_2^2)}\right)^2$$
$$+\frac{135}{16}\left(\frac{m_1m_2(m_1-m_2)}{(m_1^2+m_1m_2+m_2^2)}\right)^2\Bigg]\Bigg\}.$$

It can be consistent with Eq. (32).

Numerical results for the effects of Jupiter on the apsidal motion of the minor planets and perhaps the zodiacal dust cloud's Symmetry Surface will be addressed in a future paper.

5. Summary

We have addressed the problem of the apsidal line motion in the three-body problem and found quasi-analytical solutions to the Lagrangian solution of the three-body problem. We also stated the importance of such results on investigating the effects of the planets on the "Symmetry Surface" of the zodiacal cloud, namely determining the ascending node Ω of the symmetry surface as a function of heliocentric distance and thus discerning the role of each planet in gravitationally perturbing the orbital elements of the interplanetary dust. This will not be a trivial task given the meager information at present on the origin of the zodiacal dust, however we believe this is a first step in that direction.

It is also noteworthy to mention here that Ziglin (1976) discussed the arbitrary three-body problem in a manner where the third body has a negligibly small mass and defined quantitatively the problem for all permissible values of the parameters. In particular, the problem of critical inclinations and eccentricities was solved. He also found parameter values for which plane retrograde motions are unstable. All these results were numerical solutions of the exact equations of the three-body problem.

The series of papers of Solovaya (1972, 1974) contained analytical studies of non-restricted 3-body problems. Assuming that the angular momentum of the outer body is large enough, "new" effects appear. But no analytical results were obtained for the apsidal motion.

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APPENDIX

Assume θ_1, θ_2, θ_3 and θ are the angles which O_1X_1, O_2X_2, O_3X_3 and r make with a fixed direction in their plane, and assume that $\theta - \theta_1 = I$ and $\theta - \theta_2 = II$:

$$r_{12} = \left(\begin{bmatrix} \cos I & \sin I & 0 \\ -\sin I & \cos I & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1' \\ y_1' \\ z_1' \end{bmatrix} - \begin{bmatrix} \cos II & \sin II & 0 \\ -\sin II & \cos II & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2' \\ y_2' \\ z_2' \end{bmatrix} + \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \right)^{1/2},$$

$$\begin{aligned} \Delta = r_{12} = {} & r^2 - 2r(x_1' \cos I + y_1' \sin I + x_2' \cos II + y_2' \sin II) \\ & + \frac{1}{r^2}(x_1'^2 + y_1'^2 + z_1'^2) + (x_2'^2 + y_2'^2 + z_2'^2) \\ & - 2(x_1'x_2' + y_1'y_2') \cos I \cos II - 2(x_2'y_1' - x_1'y_2') \sin I \cos II \\ & - 2(x_1'y_2' + x_2'y_1') \sin I \cos I - 2(x_1'x_2' - y_1'y_2') \sin I \sin II, \end{aligned}$$

$$\begin{aligned} \frac{1}{r_{12}} = \frac{1}{\sqrt{\Delta}} = \frac{1}{r} \bigg[& 1 - \frac{2}{r}(x_1' \cos I + y_1' \sin I + x_2' \cos II + y_2' \sin II) \\ & + \frac{1}{r^2}(x_1'^2 + y_1'^2 + z_1'^2) + (x_2'^2 + y_2'^2 + z_2'^2) + \cdots \bigg]^{-1/2}. \end{aligned}$$

Since the odd cross product terms make no contribution to the integral $\frac{G dV_1' dV_2'}{r_{12}}$, we therefore neglect the high order terms. Thus the resultant expression will be:

$$\begin{aligned} \frac{1}{r_{12}} = 1/r \bigg[& 1 + \frac{1}{r}(x_1' \cos I + y_1' \sin I + x_2' \cos II + y_2' \sin II)^2 - \frac{1}{2r^2}(x_1'^2 + y_1'^2 + z_1'^2 + x_2'^2 + y_2'^2 + z_2'^2) \\ & + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{4}{2!}\right) \frac{1}{r^2}(x_1' \cos I + y_1' \sin I + x_2' \cos II + y_2' \sin II)^2 + \cdots \bigg], \end{aligned}$$

$$\int \frac{dV_1' dV_2'}{r} = \frac{m_1 m_2}{r},$$

$$\int \frac{dV_1' dV_2'}{r^2}(x_1' \cos I + y_1' \sin I + x_2' \cos II + y_2' \sin II) = 0,$$

For integral $-\frac{1}{2r^2} \int dV_1' dV_2' (x_1'^2 + y_1'^2 + z_1'^2 + x_2'^2 + y_2'^2 + z_2'^2)$:

$$x_1'^2 = [(1 + a_1)x_1 + d_1 y_1],$$

so

$$-\frac{1}{2r^2} \int x_1'^2 dV_1' dV_2' = -\frac{1}{2r^2}(1 + 2a_1) * \frac{1}{5} m_1 R_1^2 * m_2 = -\frac{R_1^2}{10r^2}(1 + 2a_1) m_1 m_2.$$

This is the same as the integral of terms involving $x_1'^2 \cos^2 I$, $x_1'y_1' \cos I \sin I$, $y_1'^2 \sin^2 I$, as well as other similar ones. So, the integral of potential energy due to the mutual attraction is:

$$\begin{aligned} V_{12} = {} & -\frac{Gm_1m_2}{r} - \frac{Gm_1m_2R_1^2}{10r^3}(1 + 2a_1) - \frac{Gm_1m_2R_1^2}{10r^3}(1 + 2b_1) - \frac{Gm_1m_2R_1^2}{10r^3}(1 + 2c_1) \\ & - \frac{Gm_1m_2R_2^2}{10r^3}(1 + 2a_2) - \frac{Gm_1m_2R_2^2}{10r^3}(1 + 2b_2) - \frac{Gm_1m_2R_2^2}{10r^3}(1 + 2c_2) \\ & + \frac{3Gm_1m_2R_1^2}{10r^3}(1 + 2a_1) \cos^2 I + \frac{3Gm_1m_2R_1^2}{10r^3}(1 + 2b_1) \sin^2 I + \frac{3Gm_1m_2}{5r^3} d_1 \sin 2I \\ & + \frac{3Gm_1m_2R_2^2}{10r^3}(1 + 2a_2) \cos^2 II + \frac{3Gm_1m_2R_2^2}{10r^3}(1 + 2b_2) \cos^2 II + \frac{3Gm_1m_2}{5r^3} d_2 \sin 2II. \end{aligned}$$

By using $a_1 + b_1 + c_1 = 0$ and $a_2 + b_2 + c_2 = 0$, we can simplify the result to:

$$V_{12} = -\frac{Gm_1m_2}{r} \left[1 + \frac{3R_1^2}{5r^2}(a_1 \cos^2 I + b_1 \sin^2 I + d_1 \sin 2I) + \frac{3R_1^2}{5r^2}(a_2 \cos^2 II + b_2 \sin^2 II + d_2 \sin 2II) \right].$$

Using the symbol $f_1 = a_1 - b_1$, we can get:

$$V_{12} = -\frac{Gm_1m_2}{r} \left[1 + \frac{3R_1^2}{10r^2}(-c_1 + f_1 \cos 2I + 2d_1 \sin 2I) + \frac{3R_1^2}{10r^2}(-c_2 + f_2 \cos 2II + 2d_2 \sin 2II) \right].$$

Appendix A

$$\cos A = \frac{r_1^2 + r_2^2 - r^2}{2r_1r_2} = \frac{(r_1^2 + r_2^2 - r^2)(r_3^2 + r_1^2 - r^2) - \sqrt{-r^4 + 2(r_2^2 + r_3^2)r^2 - (r_2^2 - r_3^2)^2}\sqrt{-r^4 + 2(r_1^2 + r_3^2)r^2 - (r_1^2 - r_3^2)^2}}{4r_1r_2r_3^2} \tag{A.1}$$

$$r^2 = \frac{(r_1^2 + r_2^2 + r_3^2) \pm \sqrt{(r_1^2 + r_2^2 + r_3^2) - 4((r_1^4 + r_2^4 + r_3^4) - (r_1^2r_2^2 + r_2^2r_3^2 + r_3^2r_3^2))}}{2}$$
$$r^2 = \frac{(r_1^2 + r_2^2 + r_3^2) \pm \sqrt{3}\sqrt{2(r_1^2r_2^2 + r_2^2r_3^2 + r_3^2r_3^2) - (r_1^4 + r_2^4 + r_3^4)}}{2}. \tag{A.2}$$

$$r^2 = \frac{2(m_1^2 + m_2^2 + m_3^2) + (m_1m_2 + m_2m_3 + m_3m_1) \pm 3(m_1m_2 + m_2m_3 + m_3m_1)}{2} * S'^2,$$
$$r^{(1)} = (m_1 + m_2 + m_3)S', \tag{A.3}$$
$$r^{(2)} = \sqrt{m_1^2 + m_2^2 + m_3^2 - m_1m_2 - m_2m_3 - m_3m_1}\,S'.$$

$$F'_x = -\frac{\partial U'}{\partial x_1} = -\frac{6}{5}Gm_1\frac{(a_1x_1 + d_1y_1)}{R_1^3},$$
$$F'_y = -\frac{\partial U'}{\partial y_1} = -\frac{6}{5}Gm_1\frac{(b_1y_1 + d_1x_1)}{R_1^3},$$
$$F'_z = -\frac{\partial U'}{\partial z_1} = -\frac{6}{5}Gm_1\frac{(c_1z_1)}{R_1^3},$$
$$\mathbf{F}'_r * \mathbf{r}_1 = \frac{1}{3}(\mathbf{F}'_x * \xi + \mathbf{F}'_y * \eta + \mathbf{F}'_z * \zeta) \tag{A.4}$$
$$= \frac{1}{3} * \left(-\frac{6}{5}Gm_1\right)\frac{(a_1x_1 + d_1y_1)^2 + (b_1y_1 + d_1x_1)(b_1y_1 + e_1x_1) + (c_1^2z_1^2)}{R_1^3},$$
$$V_d = \sum \Delta m_1\mathbf{F}'_r * \Delta\mathbf{r} = \frac{2}{25}Gm_1^2\frac{(a_1^2 + b_1^2 + c_1^2 + 2d_1^2)}{R_1}.$$

$$M = \frac{m_1(m_2 + m_3) + m_2(m_3 + m_1) + m_3(m_1 + m_2) + 3m_1m_2m_3}{(m_1 + m_2 + m_3)^2}\frac{m_1m_2 + m_2m_3 + m_3m_1}{m_1 + m_2 + m_3}. \tag{A.5}$$

$$\frac{d}{dt}[(2 - c_1)^2\dot\theta_1 + f_1(2\dot d_1 + f_1\dot\theta_1) - 2d_1(\dot f_1 - 2\dot d_1\dot\theta_1)] = \frac{3G\langle\sqrt{3}m_2m_3(m_2 - m_3)f_1 - 2(m_2 + m_3)(2m_2^2 - m_2m_3 + 2m_3^2)d_1\rangle}{4r^3(m_2^2 + m_2m_3 + m_3^2)}. \tag{A.6}$$

$$\frac{d}{dt}[(1 - c_1)\dot\theta_1] = \frac{3G\langle\sqrt{3}m_2m_3(m_2 - m_3)f_1 - 2(m_2 + m_3)(2m_2^2 - m_2m_3 + 2m_3^2)d_1\rangle}{4r^3(m_2^2 + m_2m_3 + m_3^2)}. \tag{A.7}$$

$$(1 - c'_1)\ddot\theta'_1 - \dot c'_1\dot\theta = \frac{3G\langle\sqrt{3}m_2m_3(m_2 - m_3)f'_1 - 2(m_2 + m_3)(2m_2^2 - m_2m_3 + 2m_3^2)d'_1\rangle}{r^3(m_2^2 + m_2m_3 + m_3^2)}, \tag{A.8}$$

$$0 = \frac{\partial u}{\partial\theta}\bigg|\,\theta = \theta_p = \left[A - \int_0^\theta(\ldots)\right](-\sin\theta_p) + [(\ldots)]\cos\theta_p,$$
$$\theta_p = \tan^{-1}\left(\frac{\int_0^{2\pi}(\delta u^2 + \delta' u^5)\cos\phi\,d\phi}{\frac{AMh^2}{G(m_1m_2+m_2m_3+m_3m_1)} - \int_0^{2\pi}(\delta u^2 + \delta' u^5)\sin\phi\,d\phi}\right). \tag{A.9}$$