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Ranging algebraically with more observations than unknowns

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In the recently developed *Spatial Reference System* that is designed to check and control the accuracy of the three-dimensional coordinate measuring machines and tooling equipment (Metronom US., Inc., Ann Arbor: `http://www.metronomus.com`), the coordinates of the edges of the instrument are computed from distances of the bars. The use of distances in industrial application is fast gaining momentum just as in *Geodesy* and in *Geophysical applications* and thus necessitating efficient algorithms to solve the *nonlinear distance equations*. Whereas the ranging problem with minimum known stations was considered in our previous contribution in the same Journal, the present contribution extends to the case where one is faced with *many distance observations* than *unknowns* (*overdetermined case*) as is usually the case in practise. Using the *Gauss-Jacobi Combinatorial approach*, we demonstrate how one can proceed to position without reverting to *iterative* and *linearizing* procedures such as *Newton's* or *Least Squares* approach.
Key words: Overdetermined planar ranging, overdetermined three-dimensional ranging, Gauss-Jacobi combinatorial algorithm, Groebner basis, Multipolynomial resultant, Least Squares.

1. Introduction

In Awange *et al.* (2003), we highlighted the importance of ranging problem and presented algebraic procedures of *reduced Groebner basis* and *Multipolynomial resultant* approaches for solving the *minimum ranging problem* directly or explicitly in a closed form. In practise however, one is often faced with a situation where more observations than unknowns exist. In such cases, the usual practise is often to rely on *linearization* or *iterative* procedures.

The bottleneck to the procedures of *linearization* and *iterative* of relying heavily on the initial starting values to obtain global minimum and faster convergence has been treated in recent works of Xu (2002, 2003). In general, the problem of nonlinear adjustment has been considered by among others Teunissen (1990), Grafarend and Schaffrin (1989, 1991) who extend the work of Krarup (1982) on nonlinear adjustment with respect to geocentric interpretation and Guolin (2000) who presents a procedure that uses F-distributions to test whether the nonlinear model can be linearized or not. An extensive analysis of the nonlinear problem with an elaborate literature review has been presented by Lohse (1994) and Mautz (2001).

The common features with the non-algebraic approaches in solving nonlinear problems is that they all have to do with some starting values, linearization of the observation equations and iterations. Although the issue of approximate starting values has been addressed in the works of Xu (2002, 2003), the algebraic (approach e.g. Awange, submitted) enjoys the advantage that all the requirements listed above for non-algebraic approaches are immaterial. The nonlinear problem is solved in an exact form with linearization permitted only during the formation of the variance-covariance matrix to generate the weight matrix of the pseudo-observations. No starting values, linearization of the observation equations, iterations and convergence conditions are required. The only requirement is to be able to solve in a closed (exact) form systems of nonlinear equations, a condition already presented in Awange *et al.* (2003). Other added advantages of the algebraic approach is that during the solution of the combinatorial subsets, any presence of outlying observations can be diagnosed (e.g. Awange, in press) and also that it provides an independent approach that can be used to control the non-algebraic procedures.

The present contribution extends on the work of Awange *et al.* (2003) by employing the *Gauss-Jacobi Combinatorial* approach presented in Awange and Grafarend (2003, in press) to solve without *linearization* or *iteration* the *overdetermined ranging problem*. We employ the *algebraic approaches* presented in Awange *et al.* (2003) as computing engine. The contribution provides efficient techniques based on algebra that have already been applied in Geodesy as evidenced in the work of Awange (2002) and Awange and Grafarend (2002a, 2002b, 2003). The technique could also be applied in Geophysics and also in Industrial applications e.g. in the work of Jurisch *et al.* (2003).

We organize the present contribution as follows; in Section 2, we present a summary of the *Gauss-Combinatorial algorithm* while in Section 3, the overdetermined two-dimensional ranging problem is solved. Section 4 considers the three-dimensional case, while Section 5 concludes the study.

2. Gauss-Jacobi Algorithm

The Gauss-Jacobi combinatorial algorithm named after C. F. Gauss (Awange and Grafarend 2003, appendix A) and C. G. I. Jacobi operates in *three* phases. In the *first* phase, one forms *minimal combinations* of the nonlinear equations from the observation sample that are solved in a closed form using the *Groebner basis* or *Multipolynomial resultant* algebraic techniques discussed in Awange *et al.* (2003) to obtain the desired combinatorial solutions. The net result is that one ends with pseudo-observations, which are within the solution space of the desired values. This first phase in essence *projects a nonlinear* case into a *linear* case. The process of solving the minimal combinatorial subsets is akin to the Gauss-elimination technique used for solving linear system of equations.

Once the first phase is successfully carried out with the solutions of the various subsets acting as pseudo-observations, the *nonlinear variance-covariance/error propagation* has to be carried out in the *second phase* to obtain the *weight matrix* of the pseudo-observations. This then requires that the stochasticity of the initial observational sample be known in order to propagate them to the pseudo-observations.

The *final phase* entails the adjustment step, which is performed to obtain the *barycentric values*. Since the pseudo-observations are linearly independent, the *special linear Gauss-Markov model* (Awange and Grafarend, 2003, definition 2-1) is employed.

Stepwise, the *Gauss-Jacobi Combinatorial algorithm* discussed in detail in Awange and Grafarend (2003) operates as following:

Step 1: Given an overdetermined system with n observations in m unknowns, from the n observations form the

$$k(\text{no. of combinations}) = \frac{n!}{m!(n-m)!} \tag{2-1}$$

minimal combination that comprise m equations that are to be solved in closed form using the *Groebner basis* or *Multipolynomial resultant* algebraic techniques.

Step 2: Solve each set of m equations from *Step 1* above using either *Groebner basis* or *Multipolynomial resultant* algebraic techniques already presented in Awange *et al.* (2003).

Step 3: Perform the *nonlinear error/variance-covariance* propagation to obtain the variance-covariance matrix of the pseudo-observations obtained in the i-th combinatorial solutions of *Step 2*. Once this has been done for all the k combinatorials, a unified variance-covariance/dispersion matrix for the entire pseudo-observations is computed e.g. as in equation (3-2).

Step 4: Using the pseudo-observations of *Step 2* and the variance-covariance matrix from *Step 3*, adjust the pseudo-observations via the special *linear Gauss-Markov model*.

Example 2-1 :

The following example based on a linear case illustrates the principles behind the algorithm.
Consider a case where three linear equations have been given for the purpose of solving the two unknowns (x, y). Three possible combinations each containing two equations necessary for solving the two unknowns in a closed form can be formed. Each of the system of two linear equations is either solved by substitution, graphically or matrix form to give three pairs of solutions $\{x_{1,2}, y_{1,2}, x_{2,3}, y_{2,3}, x_{1,3}, y_{1,3}\}$. The final step now involves the adjustment of these pseudo-observations $\{x_{1,2}, y_{1,2}, x_{2,3}, y_{2,3}, x_{1,3}, y_{1,3}\}$ with the weight matrix $\boldsymbol{\Sigma}$ obtained via *nonlinear error/variance-covariance propagation*.

Extensive exposition of the *Gauss-Jacobi combinatorial* algorithm is presented in Awange and Grafarend (2003, in press).

Table 1. Distance observations to the unknown station N.

Pt. No.	Easting $x[m]$	Northing $y[m]$	s_i $[m]$
1	48177.62	6531.28	611.023
2	49600.15	7185.19	1529.482
3	49830.93	5670.69	1323.884
4	47863.91	5077.24	1206.524

3. Overdetermined Ranging

3.1 Overdetermined two-dimensional ranging

In order to solve the overdetermined 2d ranging problem, we refer to the *Gauss-Jacobi combinatorial* algorithm discussed in *Section 2*. Combinatorials are formed using (2-1) and solved in a closed form using

$$\boxed{\begin{aligned} e_2Y_0^2 + e_1Y_0 + e_0 = 0 \\ f_2X_0^2 + f_1X_0 + f_0 = 0 \end{aligned}} \tag{3-1}$$

with the coefficients as given in Awange *et al.* (2003). For each minimal combinatorial set, one also computes in the *second step* the *dispersion matrix* of the resulting pseudo-observations using Eqs. (3-8), (3-9) and (3-10) of Box (3-1) and

$$D\{\boldsymbol{x}\} = \boldsymbol{J}_x^{-1}\boldsymbol{J}_y\boldsymbol{\Sigma}_y\boldsymbol{J}_y'(\boldsymbol{J}_x^{-1})' \tag{3-2}$$

with $\boldsymbol{J}_x, \boldsymbol{J}_y$ being the partial derivatives of (3-9) and (3-10) with respect to $\boldsymbol{x}$, $\boldsymbol{y}$ respectively at the Taylor points $(\boldsymbol{\mu}_x, \boldsymbol{\mu}_y)$. The approximate values of unknown parameters $\{x_1, \ldots, x_m\} \in \boldsymbol{x}$ appearing in the Jacobi matrices $\boldsymbol{J}_x, \boldsymbol{J}_y$ are obtained from *Groebner basis* or *Multipolynomial resultants* solution of the *nonlinear system of equations* (3-5). Finally the pseudo-observation are adjusted by the use of *special linear Gauss-Markov model* in Step 3 with the unknowns estimated via **B**est **L**inear **U**niformly **U**nbiased **E**stimator **BLUUE** (Awange and Grafarend, 2003, equation 15) as

$$\hat{\boldsymbol{\xi}} = (\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{y} \tag{3-3}$$

and the regular dispersion matrix of the estimated parameters given by

$$D\{\hat{\boldsymbol{\xi}}\} = (\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A})^{-1}. \tag{3-4}$$

The procedure becomes clear once we consider as example from Kahmen and Faig (1988) next.

Box 3-1 (*error propagation for planar ranging problem*):

For the unknown point $P(X, Y) \in \mathbb{E}^2$ of the planar ranging problem, let distances S_1 and S_2 be measured to two known points $P_1(X_1, Y_1) \in \mathbb{E}^2$ and $P_2(X_2, Y_2) \in \mathbb{E}^2$ respectively, the distance equations expressed as

$$\begin{bmatrix} S_1^2 = (X_1 - X)^2 + (Y_1 - Y)^2 \\ S_2^2 = (X_2 - X)^2 + (Y_2 - Y)^2 \end{bmatrix} \tag{3-5}$$

which we express in algebraic form as

$$\begin{bmatrix} f_1 := (X_1 - X)^2 + (Y_1 - Y)^2 - S_1^2 = 0 \\ f_2 := (X_2 - X)^2 + (Y_2 - Y)^2 - S_2^2 = 0 \end{bmatrix}. \tag{3-6}$$

On taking total differential of (3-6) we have

$$\begin{bmatrix} df_1 := 2(X_1 - X)dX_1 - 2(X_1 - X)dX + 2(Y_1 - Y)dY_1 - \\ -2(Y_1 - Y)dY - 2S_1 dS_1 = 0 \\ df_2 := 2(X_2 - X)dX_2 - 2(X_2 - X)dX + 2(Y_2 - Y)dY_2 - \\ -2(Y_2 - Y)dY - 2S_2 dS_2 = 0 \end{bmatrix} \tag{3-7}$$

which on arranging the differential vector of the unknown terms $\{X, Y\} = \{x_1, x_2\} \in \boldsymbol{x}$ on the left hand side and that of the known terms $\{X_1, Y_1, X_2, Y_2, S_1, S_2\} = \{y_1, y_2, y_3, y_4, y_5, y_6\} \in \boldsymbol{y}$ on the right hand side leads to

$$\boldsymbol{J}_x \begin{bmatrix} dX \\ dY \end{bmatrix} = \boldsymbol{J}_y \begin{bmatrix} dS_1 \\ dX_1 \\ dY_1 \\ dS_2 \\ dX_2 \\ dY_2 \end{bmatrix} \tag{3-8}$$

with

$$\boldsymbol{J}_x = \begin{bmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} \end{bmatrix} = \begin{bmatrix} -2(X_1 - X) & -2(Y_1 - Y) \\ -2(X_2 - X) & -2(Y_2 - Y) \end{bmatrix} \tag{3-9}$$

and

$$\begin{bmatrix} \boldsymbol{J}_y = \begin{bmatrix} \frac{\partial f_1}{\partial S_1} & \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial Y_1} & 0 & 0 & 0 \\ 0 & 0 & & \frac{\partial f_2}{\partial S_2} & \frac{\partial f_2}{\partial X_2} & \frac{\partial f_2}{\partial Y_2} \end{bmatrix} = \\ = \begin{bmatrix} 2S_1 & -2(X_1 - X) & -2(Y_1 - Y) & 0 & 0 & 0 \\ 0 & 0 & 0 & 2S_2 & -2(X_2 - X) & -2(Y_2 - Y) \end{bmatrix} \end{bmatrix} \tag{3-10}$$

If we consider that

$$D\{\boldsymbol{x}\} = \boldsymbol{\Sigma}_x = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{YX} & \sigma_Y^2 \end{bmatrix} \text{ and } D\{\boldsymbol{y}\} = \boldsymbol{\Sigma}_y = \begin{bmatrix} \sigma_{S_1}^2 & \sigma_{S_1X_1} & \sigma_{S_1Y_1} & \sigma_{S_1X_2} & \sigma_{S_1S_2} & \sigma_{S_1Y_2} \\ \sigma_{X_1S_1} & \sigma_{X_1}^2 & \sigma_{X_1Y_1} & \sigma_{X_1S_2} & \sigma_{X_1X_2} & \sigma_{X_1Y_2} \\ \sigma_{Y_1S_1} & \sigma_{Y_1X_1} & \sigma_{Y_1}^2 & \sigma_{Y_1S_2} & \sigma_{Y_1X_2} & \sigma_{Y_1Y_2} \\ \sigma_{S_2S_1} & \sigma_{S_2X_1} & \sigma_{S_2Y_1} & \sigma_{S_2}^2 & \sigma_{S_2X_2} & \sigma_{S_2Y_2} \\ \sigma_{X_2S_1} & \sigma_{X_2X_1} & \sigma_{X_2Y_1} & \sigma_{X_2S_2} & \sigma_{X_2}^2 & \sigma_{X_2Y_2} \\ \sigma_{Y_2S_1} & \sigma_{Y_2X_1} & \sigma_{Y_2Y_1} & \sigma_{Y_2S_2} & \sigma_{Y_2X_2} & \sigma_{Y_2}^2 \end{bmatrix} \tag{3-11}$$

we obtain with (3-8), (3-9) and (3-10) the dispersion $D\{\boldsymbol{x}\} = \boldsymbol{J}_x^{-1}\boldsymbol{J}_y\boldsymbol{\Sigma}_y\boldsymbol{J}_y'(\boldsymbol{J}_x^{-1})'$ of the unknown variables $\{X, Y\} = \{x_1, x_2\} \in \boldsymbol{x}$.

Example:

Let us consider the example of Kahmen and Faig (1988, pp. 240–241) where the coordinates of point N are to be determined from distance observations to four points P_1, P_2, P_3 and P_4 (i.e. figure 6.4.4 of Kahmen and Faig, ibid, p. 229). In preparation for adjustment, the distances are corrected and reduced geometrically to Gauss-Krueger projection and are as given in Table 1.

In order to test the *Gauss-Jacobi combinatorial* algorithm, we compute the coordinates of station N and compare it

Table 2. Position of station N computed for various combinatorials.

Combinatorial No.	Combinatorial points	x $[m]$	y $[m]$
1	1-2	48565.2783	6058.9770
2	1-3	48565.2636	6058.9649
3	1-4	48565.2701	6058.9702
4	2-3	48565.2697	6058.9849
5	2-4	48565.3402	6058.9201
6	2-5	48565.2661	6058.9731

Table 3. Position of station N after adjustments.

Approach	$x(m)$	$y(m)$	$\sigma_x(m)$	$\sigma_y(m)$	$\Delta_x(m)$	$\Delta_y(m)$
Least Squares	48565.2700	6058.9750	0.006	0.006	—	—
Gauss-Jacobi (BLUUE)	48565.2709	6058.9750	0.0032	0.0034	−0.0009	0.0000
Gauss-Jacobi (arithmetic mean)	48565.2813	6058.9650	—	—	−0.01133	0.0100

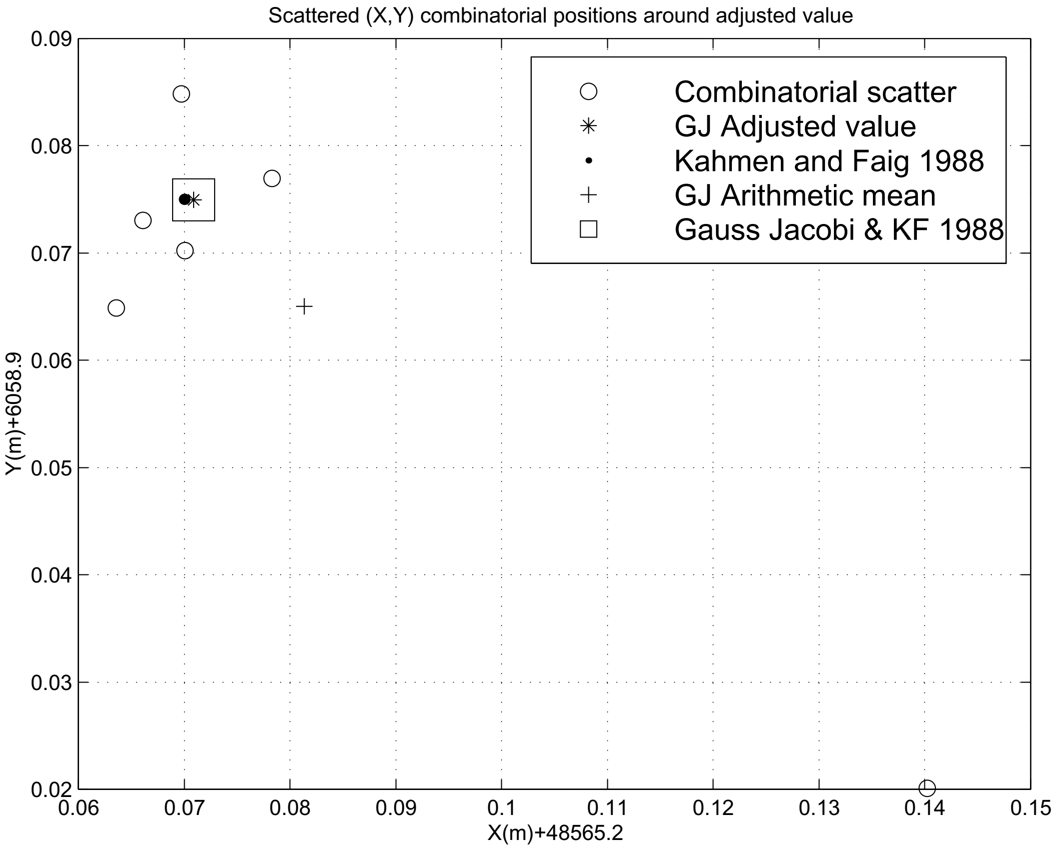

Fig. 1. Plot of the position of N from various approaches.

with the Least Squares value of Kahmen and Faig (1988, p. 242). From (2-1), six combinatorials in the minimal sense are formed with each combinatorial solved for $\{x, y\}_N$ for point N using (3-1) as discussed in Awange *et al.* (2003). The combinatorial solutions are presented in Table 2.

The barycentric coordinate of station N is now obtained either by (a) simply taking the arithmetic mean of the combinatorial solutions in columns 3 and 4 of Table 2 (an approach which does not take into account full information in terms of the variance-covariance matrix) of the pseudo-observations or (b) by using *special linear Gauss-Markov model* through the estimation by the **B**est **L**inear **U**niformly **U**nbiased **E**stimator **BLUUE** in Eq. (3-3) and the dispersion obtained by (3-4). The results are presented in Table 3 and plotted in Fig. 1. In Table 3, we present the coordinates $\{x, y\}$ of station N obtained using the Least Squares approach in Kahmen and Faig (1988), *Gauss-Jacobi combinatorial* (**BLUUE**) and the *Gauss-Jacobi combinatorial* (arith-

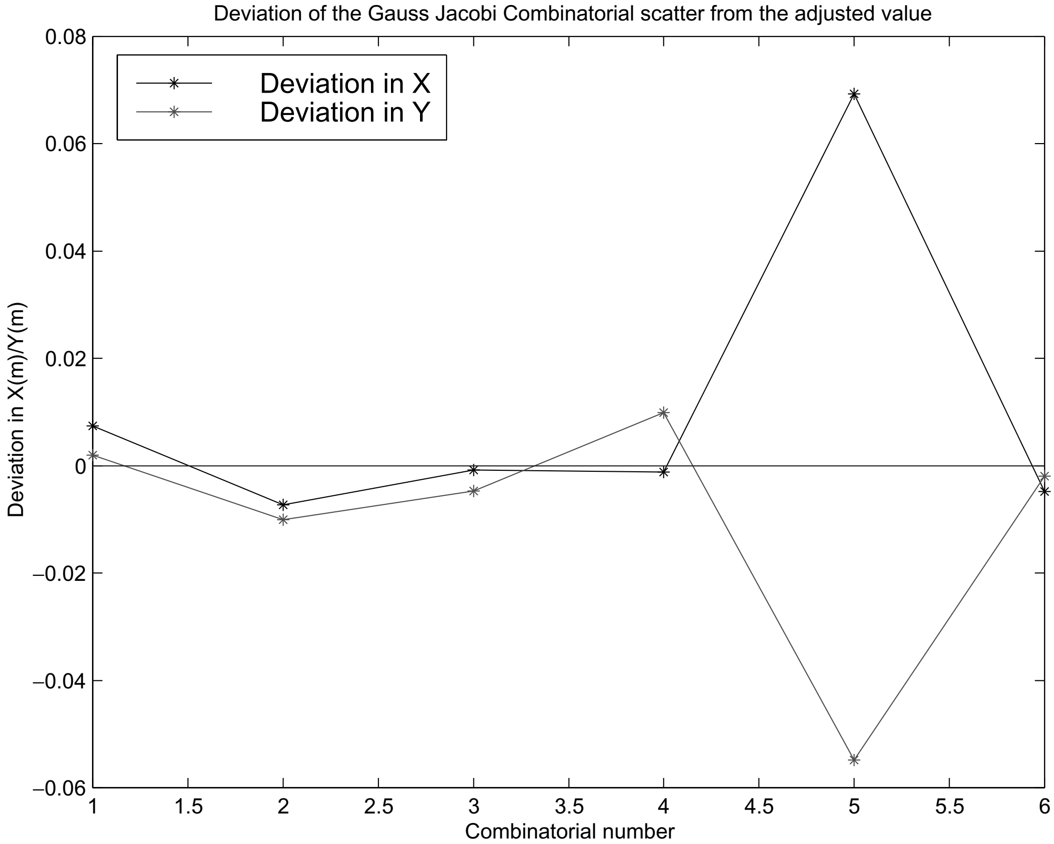

Fig. 2. Deviations of the combinatorial scatter from the BLUUE adjusted position of N.

metic mean) in columns 2 and 3 with their respective standard deviations $\{\sigma_x, \sigma_y\}$ in columns 4 and 5. In columns 6 and 7, the deviations $\{\Delta_x, \Delta_x\}$ of the computed coordinates of station N using the *Gauss-Jacobi combinatorial* (**BLUUE** and arithmetic mean) from the values of Kahmen and Faig (1988) are presented. The deviations of the exact solutions of each combination (columns 3 and 4 of Table 2) from the adjusted values of **B**est **L**inear **U**niformly **U**nbiased **E**stimator **BLUUE** (i.e. second raw of Table 3) obtained using Eq. (3-3) are plotted in Fig. 2.

From the results in Table 3 and Fig. 1, it is seen that when the full information of the observation is taken into account via the *nonlinear error/variance-covariance* propagation and the parameters estimated via BLUUE for the linear Gauss-Markov model in the final step for the barycentric coordinates, the Gauss-Jacobi combinatorial algorithm gives the same results as *Least Squares* adjustment (from Kahmen and Faig, 1988). In addition to giving the barycentric coordinates, the *Gauss-Jacobi algorithm* can accurately pin point a poor combinatorial geometry (e.g. combination 5) although this is taken care of through weighting. Figure 1 shows the combinatorial scatter denoted by $\{\circ\}$ and the *Gauss-Jacobi combinatorial* adjusted value with $\{*\}$. Least Squares estimation from Kahmen and Faig (1988) by $\{\bullet\}$ and the arithmetic mean by $\{+\}$. One notes that the estimates from Gauss-Jacobi's BLUUE $\{*\}$ and Least Squares estimation from Kahmen and Faig (1988) almost coincide. In the Figure, both estimates are encircled by $\{\Box\}$ for clarity purposes.

These results indicate the capability of the *Gauss-Jacobi combinatorial algorithm* to solve *overdetermined* planar ranging problems.

3.2 Overdetermined three-dimensional ranging

Having solved the overdetermined planar ranging problem in Section 3.1, we extend the use of the *Gauss-Jacobi combinatorial algorithm* to solve the overdetermined three-dimensional ranging problem in this section. An example based on the test network *"Stuttgart Central"* in *Fig.* 3 is considered.

Example

The test network *"Stuttgart Central"* in *Fig.* 3 consists of distance observations from station K_1 to seven other stations. Desired are the three-dimensional coordinates X, Y, Z of the unknown point K_1 obtained by solving three nonlinear distance observation equations in closed form discussed in Awange *et al.* (2003).

From Fig. 3 and using (2-1), 35 combinatorial subsets are formed whose systems of *nonlinear distance equations* are solved for the position X, Y, Z of the unknown point K_1 in closed form using either *Groebner basis* approach derived equations (e.g. Awange *et al.*, 2003, *box* 3-4)

$$Y = \frac{\{(a_{12}c_{02} - a_{02}c_{12})Z + a_{12}f_{02} - a_{02}f_{12}\}}{(a_{02}b_{12} - a_{12}b_{02})} \tag{3-12}$$

and

$$X = \frac{-(b_{12}Y + c_{12}Z + f_{12})}{a_{12}}$$

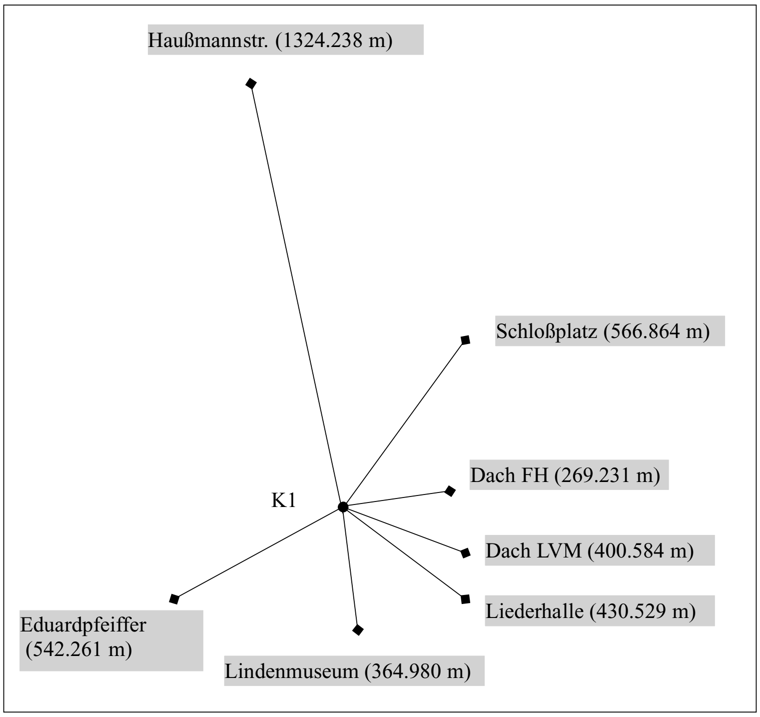

Fig. 3. Test network *"Stuttgart Central"*.

or

$$X = \frac{\{(b_{02}c_{12} - b_{12}c_{02})Z + b_{02}f_{12} - b_{12}f_{02}\}}{(a_{02}b_{12} - a_{12}b_{02})} \tag{3-13}$$

or *Multipolynomial resultant* approach derived equations (e.g. Awange *et al.*, 2003, *box* 3-5)

$$X = \frac{\{(b_{12}c_{02} - b_{02}c_{12})Z + b_{12}f_{02} - b_{02}f_{12}\}}{(b_{02}a_{12} - b_{12}a_{02})} \tag{3-14}$$

and

$$Y = \frac{\{(a_{12}c_{02} - a_{02}c_{12})Z + a_{12}f_{02} - a_{02}f_{12}\}}{(a_{02}b_{12} - a_{12}b_{02})}. \tag{3-15}$$

35 different positions $X, Y, Z|_P$ of the same point P totalling to 105 (35 × 3) values of X, Y, Z which are treated as pseudo-observations are obtained. One then proceeds in two steps as follows:

Step 1: From the 35 combinatorials obtained using (2-1), solve for X, Y, Z in close form using either (3-12) and (3-12) or (3-14) and (3-15).

Step 2: (Error propagation to determine the dispersion matrix $\mathbf{\Sigma}$ based on linearized approximation):

The variance-covariance matrix is computed for each of the combinatorial set $j = 1, \ldots, 35$ using error propagation. The closed form observational equations for the first combinatorial subset $j = 1$ (i.e. tetrahedron $PP_1P_2P_3$) Awange *et al.* (2003) are written algebraically as

$$\begin{cases} f_1 := (X_1 - X)^2 + (Y_1 - Y)^2 + (Z_1 - Z)^2 - S_1^2 \\ f_2 := (X_2 - X)^2 + (Y_2 - Y)^2 + (Z_2 - Z)^2 - S_2^2 \\ f_3 := (X_3 - X)^2 + (Y_3 - Y)^2 + (Z_3 - Z)^2 - S_3^2 \end{cases} \tag{3-16}$$

where $S_i^j \, |i \in \{1, 2, 3\} \mid j = 1$ are the distances between known GPS stations $P_i \in \mathbb{E}^3 \, |i \in \{1, 2, 3\}$ of the test network *"Stuttgart Central"* and the unknown GPS point $P \in \mathbb{E}^3$ for first combination set $j = 1$. With (3-9) and (3-10) we have the Jacobi matrices respectively as

$$\boldsymbol{J}_x = \begin{bmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial Z} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial Z} \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial Z} \end{bmatrix} = \begin{bmatrix} -(X_1 - X) & -(Y_1 - Y) & -(Z_1 - Z) \\ -(X_2 - X) & -(Y_2 - Y) & -(Z_2 - Z) \\ -(X_3 - X) & -(Y_3 - Y) & -(Z_3 - Z) \end{bmatrix} \tag{3-17}$$

and

$$\boldsymbol{J}_y = \begin{bmatrix} \frac{\partial f_1}{\partial S_1} & \frac{\partial f_1}{\partial S_2} & \frac{\partial f_1}{\partial S_3} & \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial Y_1} & \frac{\partial f_1}{\partial Z_1} & \frac{\partial f_1}{\partial X_2} & \frac{\partial f_1}{\partial Y_2} & \frac{\partial f_1}{\partial Z_2} & \frac{\partial f_1}{\partial X_3} & \frac{\partial f_1}{\partial Y_3} & \frac{\partial f_1}{\partial Z_3} \\ \frac{\partial f_2}{\partial S_1} & \frac{\partial f_2}{\partial S_2} & \frac{\partial f_2}{\partial S_3} & \frac{\partial f_2}{\partial X_1} & \frac{\partial f_2}{\partial Y_1} & \frac{\partial f_2}{\partial Z_1} & \frac{\partial f_2}{\partial X_2} & \frac{\partial f_2}{\partial Y_2} & \frac{\partial f_2}{\partial Z_2} & \frac{\partial f_2}{\partial X_3} & \frac{\partial f_2}{\partial Y_3} & \frac{\partial f_2}{\partial Z_3} \\ \frac{\partial f_3}{\partial S_1} & \frac{\partial f_3}{\partial S_2} & \frac{\partial f_3}{\partial S_3} & \frac{\partial f_3}{\partial X_1} & \frac{\partial f_3}{\partial Y_1} & \frac{\partial f_3}{\partial Z_1} & \frac{\partial f_3}{\partial X_2} & \frac{\partial f_3}{\partial Y_2} & \frac{\partial f_3}{\partial Z_2} & \frac{\partial f_3}{\partial X_3} & \frac{\partial f_3}{\partial Y_3} & \frac{\partial f_3}{\partial Z_3} \end{bmatrix}$$

$$= \begin{bmatrix} S_1 & 0 & 0 & j_{14} & j_{15} & j_{16} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & S_2 & 0 & 0 & 0 & 0 & j_{27} & j_{28} & j_{29} & 0 & 0 & 0 \\ 0 & 0 & S_3 & 0 & 0 & 0 & 0 & 0 & 0 & j_{10} & j_{11} & j_{12} \end{bmatrix} \tag{3-18}$$

where

$$\begin{bmatrix} j_{14} = -(X - X_1), & j_{15} = -(Y_1 - Y), & j_{16} = -(Z_1 - Z) \\ j_{27} = -(X_2 - X), & j_{28} = -(Y_2 - Y), & j_{29} = -(Z_2 - Z) \\ j_{10} = -(X_3 - X), & j_{11} = -(Y_3 - Y), & j_{12} = -(Z_3 - Z). \end{bmatrix}$$

The values $\{X, Y, Z\}$ appearing in the Jacobi matrices $\boldsymbol{J}_x, \boldsymbol{J}_y$ are obtained from the closed form solution using either (3-

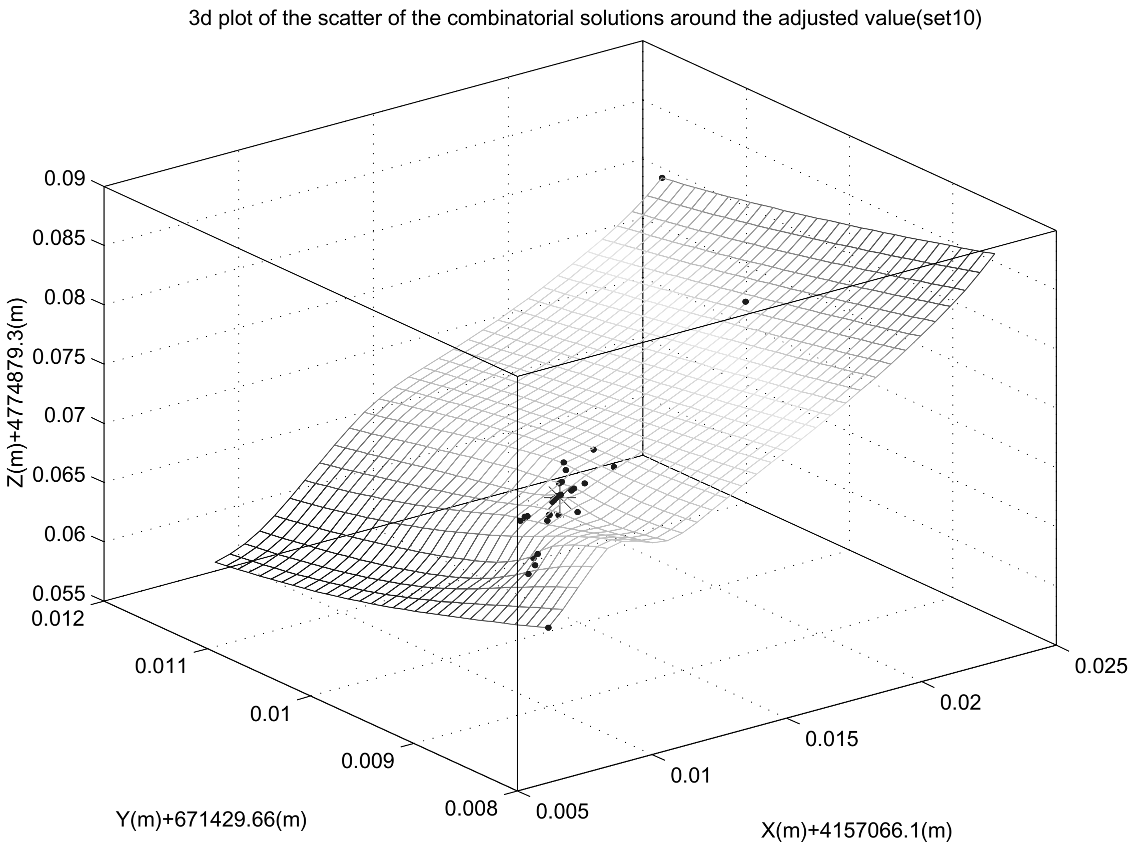

Fig. 4. Scatter of combinatorial solutions.

12) and (3-12) or (3-14) and (3-15). From the dispersion matrix $\mathbf{\Sigma}_y$ of the vector of observations $\boldsymbol{y}$ and with (3-17) and (3-18) forming $\boldsymbol{J} = \boldsymbol{J}_x^{-1}\boldsymbol{J}_y$, the variance-covariance matrix $\mathbf{\Sigma}_x$ is finally obtained from (3-2) as

$$\begin{bmatrix} \sigma_X^2 & \sigma_{XY} & \sigma_{XZ} \\ \sigma_{YX} & \sigma_Y^2 & \sigma_{YZ} \\ \sigma_{ZX} & \sigma_{ZY} & \sigma_Z^2 \end{bmatrix}$$

$$= \boldsymbol{J} \begin{bmatrix} \sigma_{S_1}^2 & \sigma_{S_1S_2} & \sigma_{S_1S_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma_{S_2S_1} & \sigma_{S_2}^2 & \sigma_{S_2S_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma_{S_3S_1} & \sigma_{S_3S_2} & \sigma_{S_3}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{X_1}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{Y_1}^2 & 0 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{Z_1}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{X_2}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{Y_2}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{Z_2}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{X_3}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{Y_3}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{Z_3}^2 \end{bmatrix} \boldsymbol{J}' \tag{3-19}$$

with the 3×3 elements of $\mathbf{\Sigma}_y$ on the right hand side of (3-19) given from error propagation. The variance-covariance matrix computed as explained above is obtained for every combinatorial set $j = 1, \ldots, 35$. Given $\boldsymbol{J}_i = \boldsymbol{J}_{x_i}^{-1}\boldsymbol{J}_{y_i}$ from the i-th combination and $\boldsymbol{J}_j = \boldsymbol{J}_{x_j}^{-1}\boldsymbol{J}_{y_j}$ from the j-th combination, the correlation between the i-th and j-th combination is given by

$$\mathbf{\Sigma}_{ij} = \boldsymbol{J}_j \mathbf{\Sigma}_{y_j y_i} \boldsymbol{J}_i' \tag{3-20}$$

Finally we obtained the dispersion matrix $\mathbf{\Sigma}$ from the sub-matrices variance-covariance matrix for the individual combinatorials $\mathbf{\Sigma}_1, \mathbf{\Sigma}_2, \mathbf{\Sigma}_3, \ldots, \mathbf{\Sigma}_k$ (where k is the number of combinations) obtained via (3-2) and the correlations between combinatorials obtained from (3-20) as

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Sigma}_1 & \mathbf{\Sigma}_{12} & . & . & . & \mathbf{\Sigma}_{1k} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_2 & & . & . & \mathbf{\Sigma}_{2k} \\ . & & \mathbf{\Sigma}_3 & & & \\ . & & & . & & \\ . & & & & . & \\ \mathbf{\Sigma}_{k1} & . & & . & . & \mathbf{\Sigma}_k \end{bmatrix} \tag{3-21}$$

for the entire k combinations.

Step 3: (Rigorous adjustment of the combinatorial solution points in a polyhedron):

For each of the 35 computed coordinates of point K1 in *Fig.* 3 in Step 2, we write the observation equations as

$$\begin{bmatrix} X^j = X + \varepsilon_X^j \,|, j \in \{1, 2, 3, 4, 5, 6, 7, \ldots, 35\} \\ Y^j = Y + \varepsilon_Y^j \,| j \in \{1, 2, 3, 4, 5, 6, 7, \ldots, 35\} \\ Z^j = Z + \varepsilon_Z^j \,|, j \in \{1, 2, 3, 4, 5, 6, 7, \ldots, 35\}. \end{bmatrix} \tag{3-22}$$

With the values $\{X^j, Y^j, Z^j\}$ treated as pseudo-observation and placed in the vector of observation $\boldsymbol{y}$, the coefficients of the unknown position $\{X, Y, Z\}$ being placed in the coefficient matrix $\boldsymbol{A}$ and $\boldsymbol{x}$ comprising the vector of unknowns

Table 4. Position of station K_1 computed by *Gauss-Jacobi combinatorial algorithm*.

Exp No.	$X(m)$	$Y(m)$	$Z(m)$	σ_X	σ_Y	σ_Z
1	4157066.1121	671429.6694	4774879.3697	0.00005	0.00001	0.00005

Table 5. Deviation of the computed position of K_1 in Table 2 from the real measured GPS value.

Exp No.	$\Delta X(m)$	$\Delta Y(m)$	$\Delta Z(m)$
1	-0.0005	-0.0039	0.0007

$\{X, Y, Z\}$, The solution is obtained via (3-3) and the dispersion of the estimated parameters through (3-4).

In the experiment above the computed position of point K_1 using the *Gauss-Jacobi combinatorial* approach is given in Table 4 with the deviation of the *Gauss-Jacobi combinatorial* solutions from the *true (measured) GPS value* given in Table 5. Figure 4 indicates the plot of the combinatorial scatter $\{\bullet\}$ around the adjusted values $\{*\}$.

4. Conclusion

For problems that require the solution of overdetermined ranging, and whose initial starting values are not known such as in photogrammetry, the *Gauss-Jacobi combinatorial algorithm* offers an alternative approach provided the full information of the underlying observation is taken into consideration via the *nonlinear variance-covariance/error propagation*. The advantage of the *Gauss-Jacobi combinatorial* being that no starting values, linearization or iterations is required as is the case with other procedures. Outlying combinations and observations are also identifiable. With high processing computers currently available, the issue of many combinatorials formed as a result of large observations is immaterial nor is the computing time. Routines are written that repeatedly execute the desired task once the statement that executes the combination has been written.

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