DC railways and the magnetic fields they produce—the geomagnetic context

F. J. Lowes

School of Chemistry, Newcastle University, Newcastle upon Tyne, NE1 7RU, UK

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DC electric railways produce magnetic fields, not only from the intended traction currents, but also from unintended earth-leakage currents; these fields, particularly those from the leakage currents, are becoming an increasing problem for geomagneticians. This paper introduces the relevant properties of DC-railway traction-power circuits, and the various ways in which earth-leakage currents are produced, and discusses models of how these leakage currents vary along the track and with train position. It describes the geometry of the resultant magnetic fields, and gives the formal algebra for calculating the magnetic field when these leakage currents are known, but also suggests some simple approximations that could be used when the current distribution is not known in detail. This paper also summarises previous relevant papers.

Key words: DC railways, earth leakage, geomagnetic field, leakage current, magnetic field, magnetic observatory.

1. Introduction

Ever since DC electric trams and trains were introduced in the 1880s, geophysicists around the world have been complaining about the interference they produce. Currents leaking to the ground from the systems produce direct effects over tens of kilometres on telluric measurements (see e.g. Kovalevskiy *et al.*, 1961; Pádua *et al.*, 2002), and have been used as a telluric source to monitor ground resistivity (Tanbo *et al.*, 2003). They can also produce ULF fields looking like pulsations (e.g. Jones and Kelly, 1966; Fraser-Smith, 1981; Egbert *et al.*, 2000), and indirectly add noise to other measurements (e.g. at the CERN LEP accelerator, Bravin *et al.*, 1998, and at a broad-band vertical seismic sensor at Stuttgart, Forbriger, 2007).

In this paper I am concerned only with the effect on quasi-DC geomagnetic field measurements. The trains themselves produce local magnetic fields (see e.g. Chadwick and Lowes, 1998), but these are significant only when observing on or near the train and will not be discussed here. Close to the railway the magnetic field produced by the traction current flowing to and from the train is significant, but at larger distances the dominant field is that produced by currents leaking from the railway track into the ground; it is these two sources of field that will be considered.

Most of the earlier papers published by observers were restricted to a description of the field in terms of its overall 'noise' characteristics. Rössiger (1942) was probably the first to associate individual magnetic field signatures with individual train movements. Dupouy (1950) was probably the first to try to model the leakage current distribution and the resultant magnetic field. Then there were occasional papers until Tokumoto and Tsunomura (1984), followed by a 20-year gap. See Appendix A for a discussion of the relevant literature. However, as the instruments at magnetic observatories become more sensitive, and DC railways proliferate, there has been increasing interest in the magnetic fields produced by the railway earth-leakage currents. Unfortunately the recent paper by Georgescu *et al.* (2002) used a non-feasible model of the leakage-current distribution. And while Pirjola *et al.* (2007) gave comprehensive algebra for the calculation of the resulting magnetic field, it used the Georgescu *et al.* model; also the simple geometry of the fields was obscured by the algebra.

The relevant properties of railway systems can vary widely, and to be able to use any leakage model the reader will first have to determine what these properties are for his local system. One aim of this paper is to give a geomagnetician sufficient knowledge about the working of DC train systems that he can discuss the relevant properties with the railway engineers. So I first describe the relevant features of the railway electrical power-supply circuit. Then in Section 3 I discuss the factors that determine the rail-to-ground leakage-current distribution, and in Section 4 give approximate and exact expressions for this leakage for some simple ideal situations.

Section 5 then describes the geometry of the magnetic field produced by the different currents, and shows how these fields can be calculated for a given distribution of leakage currents. However in many situations this current distribution will not be known well enough to warrant such a detailed calculation, so in the discussion I suggest simple approximate calculations for estimating the magnitude of the maximum likely magnetic field, once the main design features of the railway system are known. Appendix A summarises the relevant geomagnetic literature.

Because to engineers and physicists the word 'earth' has the underlying connotation of zero potential, in this paper 'earth' is used only when a specific low-resistance connection is intended (except for its use in the term 'earth-leakage'), and 'ground' is used to describe the material through which the stray currents are passing; to maintain these currents there must be finite, if small, local potential gradients.

2. DC-powered Railways

DC powered rail vehicles are used on both long-distance and commuter railways; in the latter case they are sometimes called Light Rapid Transits (LRTs) or Metros, or Trams when street running, but for simplicity I will call them all trains. The trains run on a track consisting of two steel 'running rails'. I am concerned only with trains that obtain their power from a stationary DC substation, and not, for example, with trains powered by diesel-electric locomotives that generate the current on the train itself, or with AC-powered railways where the rectification from AC to DC is on the train.

2.1 The power-supply system

The railway traction current will be supplied by rectifiers at DC substations. The behaviour of these rectifiers is quite complicated, but for the purpose of this paper it is sufficient to use the fairly crude approximation that, in their normal working range, they have a constant internal resistance (typically of the order of 0.03 Ω). Each substation will usually have two or more rectifiers, normally working in parallel. For simplicity from now on I will talk about only 'substations' and their 'internal resistance'.

To carry the current from the substation to and from the train, two conductors are needed. One of these conductors is either an overhead copper wire, or a 'third rail' on or near the ground. For simplicity I will discuss only the overhead system. The height of the wire above the ground is typically in the range 4.5–5.5 m; in numerical examples I will assume a vertical wire/track separation of 5.0 m. (For a third-rail system the equivalent separation is mainly horizontal, and much smaller; the necessary changes to the algebra of Section 5 are obvious.) With very rare exceptions, this overhead conductor is the positive conductor; it is convenient to think of the overhead as 'feeding' current to the train.

In the vast majority of systems the current is returned to the substation through one or both of the running rails, using some of the train wheels as a rolling contact. (One of the few exceptions is the London Underground, which has an insulated 'fourth rail' with a sliding contact.) It is the leakage of this return current from the running rails into the ground that causes most of the problems for workers in geomagnetism. These earth-leakage currents are also known as 'stray currents'.

Current International Standards specify nominal DC supply voltages of 750, 1500 and 3000 V. As typical loads are now of the order of 1 MW or more, this means that the traction currents are of the order of 1000 A or more. In heavily trafficked urban areas the rectifier substation spacing might be 2 km or less, but in single-track rural systems the spacing might be 20 km or more.

A typical single overhead contact wire has a resistance of about 0.17 Ω/km when new. In areas of high demand this contact wire is doubled up, connected to extra wires in the catenary support system, or connected to a parallel underground 'reinforcing feeder' cable. Even so, there are large ohmic drops in the overhead wire, and the voltage seen by the train can vary over a wide range; International Standards allow this voltage to vary between −33% and +20% (e.g. for a 750 V system, the allowable range is 500 V to 900 V). In normal running conditions the overhead wire is electrically continuous, so that there will be current sharing between adjacent substations. In double-track situations there might also be bonding between the two overheads.

Railways (though not street-running sections of Tramways) detect the presence of trains on a section of track by using 'track circuits', in which the train axles short-circuit a voltage applied between the two rails. In older systems at least one running rail has to be electrically sectioned by insulated rail joints to allow for this. On older DC systems the track circuits work at low-frequency AC, so that 'impedance bonds' (tuned inductors) can be used to give isolation between adjacent track circuits, while allowing both running rails to be used for the DC traction current. Modern systems use 'jointless' track circuits, working at higher audio and/or modulated frequencies, so that there is no need for insulated rail joints except at points and crossings. A single steel continuously-welded rail typically has a resistance of about 0.036 Ω/km. If there are two tracks, there is likely to be cross-bonding between the tracks; this reduces the along-track resistance and improves reliability and electrical safety through redundancy. Apart from the above signalling considerations, the running rails are almost always electrically continuous throughout the system. In a given situation the along-track rail resistance will depend on whether 1, 2, or 4 rails are used, but for simplicity I will refer simply to the rail, or track.

2.2 The train

Because of the wide voltage variation seen by the train, most train traction-motor control systems are designed to provide the current programmed for that particular situation, regardless of the overhead-to-rail voltage difference at that point provided it is within the system's working range. So when considering the traction current in the rail, at any one time we can replace the rectifier at the substation, and the motors etc. of the train, by a 'constant-current generator', whose output current does not depend on the voltage at its terminals; see Fig. 1.

A train travelling at a fixed speed on a level track would be working at a constant power, just sufficient to overcome rail friction and air drag. But trains need to accelerate from rest, and climb hills, and the power can be higher in these situations. The traction control system on passenger trains is often programmed to give a prescribed acceleration/time behaviour, with the driver essentially just selecting a particular programme. And on an urban passenger system having frequent stops, trains often accelerate away from a station, and then coast un-powered before braking for the next stop. So the overall traction-power/time curve will be different for different systems.

To produce a given torque (train acceleration), DC traction motors need a given current. However at low speeds the motors have a very low resistance, so the voltage applied to the motor has to be a lot less than the supply voltage. In older control systems, the necessary voltage drop

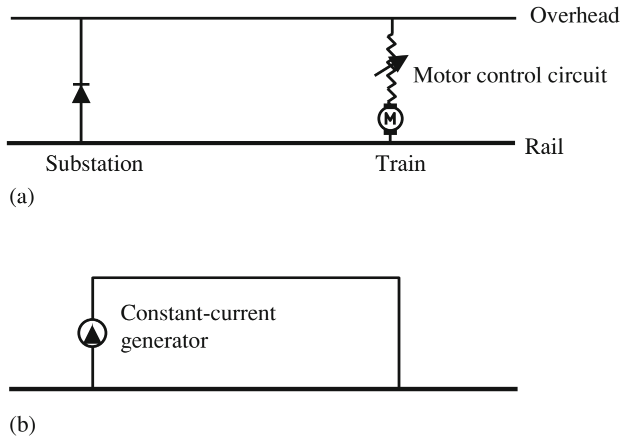

Fig. 1. (a) Rectifier at substation feeds train via overhead and rail. (b) As far as the traction current in the rail is concerned, because of the motor control circuit this is equivalent to a constant-current generator pushing current into the rail.

is produced by inserting a resistance in series. As the train speed increases, so does the back-emf in the motor, and the inserted resistance can be decreased, until it is no longer needed. In these systems the traction current in the overhead/rail circuit rises quickly to a maximum in one or two seconds, is then roughly constant for some time (typically 10 s for a Metro system), and then reduces as the motor back-emf increases. The current will be reduced further if the speed limit is reached, or switched off when the driver has enough speed to coast to the next station. However, although simple, this system is inefficient, and speed control is relatively coarse. More modern control systems are much more efficient. They use solid-state chopper electronics, followed by smoothing, and vary the on/off ratio of the chopping to obtain the required voltage for the motor. In this case, during acceleration the traction current will increase fairly smoothly from zero to a maximum. And even on DC railway systems, modern trains now use variable-frequency AC induction motors, again with solid-state electronics to produce the appropriate three-phase current over the appropriate frequency range; the traction current will have the same sort of waveform as a chopper-controlled DC motor.

Hence the shape of the traction-current waveform during acceleration will depend on the design of the train. But in all cases, the maximum traction current will be determined by the maximum traction power used by the train; for example a 1500 V system will need to provide 1000 A if the traction power is 1.5 MW.

A complication is the increasing use of regenerative braking. Braking usually involves a combination of friction and electrodynamic braking, the latter using the traction motors as dynamos to feed a load. On older systems, electrodynamic braking simply wasted the generated power in on-board braking resistors, but on modern systems regenerative braking is used. In this case the regenerated DC current is used in the train's own auxiliaries and any excess is 'pushed' back into the overhead line, provided there is another train that can take advantage of it. Only if there is no such train is the excess current diverted to the on-board braking resistors. And in a few more modern systems the substations themselves can 'accept' this regenerated DC current, pushing the equivalent AC current back into the grid. From the viewpoint of this paper, if there is regeneration there will be a reversed current in the overhead/rail system, between the regenerating train and the other train and/or substation during the braking period.

2.3 Division of current between substations

As argued above, if there is only one substation, and only one train on the track, to a good approximation the train can be considered to act as a high-internal-impedance current generator; this generator replaces the actual rectifiers, control system, and motors, and it does not matter if we think of it as being in the train or at the substation.

If we add more substations, then to the same approximation the train still draws the same current. If the train is between two substations, this current is divided between the two, the ratio of the currents being the inverse of the ratio of the resistances (overhead + substation + rail) of the two current paths; in practice this ratio is dominated by the resistance of the overhead wire. Therefore, assuming uniform overhead, to a good approximation this ratio is a linear function of the along-track train position, and is very little affected by whatever might be happening in the rails. (If there are also other, more remote, substations, then because of the finite internal resistance of the rectifiers, a small fixed fraction of the current from each direction will be supplied by the remote substations.) We can think of the train pushing its total traction current into the rail, with the relevant fraction of this current having no option but to get back to its 'own' substation. It is important to note that this division of current between the substations is determined almost entirely by the resistance of the overhead, and is hardly affected by whether or not there is earth leakage. However this division does vary as the train moves along the track.

To the good approximation that the substation/overhead/rail resistance network is linear, removing the rectifiers, and inserting a current generator between the overhead and rails at a given position will give the same distribution of *its* current between substations whether or not there are any other current generators (trains) elsewhere on the network. So, each train on the system will behave as an *independent* current generator, with the division of *its* current between the substations determined by *its* position on the track, independent of the presence of other trains. Then the overall current distribution in the overhead and rails is given by the algebraic sum of the current distributions for each train. Of course this can involve considerable cancellation of individual current contributions at some locations, but this approach does allow using a set of very simple separate calculations, rather than having to solve a set of simultaneous equations.

3. Earth-leakage Currents

3.1 Introduction

Clearly the magnetic field produced by the traction current in the loop substation-overhead-train-rails-substation will produce a significant magnetic field; this is the dominant field near the track. But at larger distances what dominates is the magnetic field due to currents that have leaked

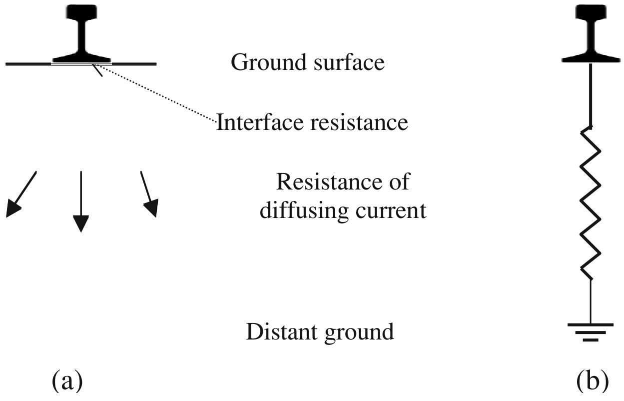

Fig. 2. Schematic picture of current leaking from a rail into the ground. (a) The real situation. (b) The railway engineer's approximation.

from the track into the ground, returning to the track and/or substation elsewhere. (Leakage from the overhead wire, or third or fourth rail, is usually insignificant, as they are much easier to insulate.)

The running rails are close to the ground, so unless precautions are taken it is very easy for them to have significant electrical contact with the ground. As a result, a proportion, sometimes quite large, of the return traction current in the rails will leak into, and elsewhere return from, the ground.

To geophysicists the near-surface ground is an electrically finitely-conducting 3-dimensional medium. The surface layers are usually soils and subsoils, the electrical conductivity of which is typically in the range 10^{-3} to 10^{-1} S/m, depending mostly on the amount and salinity of the water trapped between the grains. Deeper layers, of sedimentary or volcanic origin, are more poorly conducting, but to some extent this poorer conductivity is offset by the larger cross-sectional areas available for current flow. Whatever the spatial scale of any quasi-DC current system flowing through the ground, there must be corresponding finite, if small, potential gradients in the ground.

However the railway electrical engineer almost always approximates this near-surface ground as a highly conducting medium, in which there is essentially *zero* resistance between a region, the 'source', in which current is injected into it, and another region, the 'sink', in which the current is extracted. In their modelling of leakage currents they represent the overall effect of the region between the rail and the distant ground in terms of a leakage conductance per unit length of rail—see Fig. 2. For 'good' track this conductance might be as low as 0.01 S/km for two rails in parallel, but for leaky track might be as high as 10 S/km. The assumption is that this 'surface' rail-ground conductance is small enough (resistance large enough) that the resistance contributed by the rest of the ground between source and sink can be neglected.

Clearly this assumption cannot be true when the source and sink regions are so close together that they overlap. Fortunately such a situation occurs only where the rail voltage is about zero, where the leakage currents will be small. (Also the small separation means that the magnetic field produced is itself of small-scale, so falls-off rapidly with distance, and can probably be ignored in most situations.) For most source-sink separations the approximation is probably justified, and I will use it from now on.

3.2 Railway earthing policy

Some older railways tried to keep their track very close to earth potential; for example, in their appendix, Iliceto and Santarato (1999) said that some DC railways in Italy are "carefully earthed at discrete points about 100 m apart"; however such deliberate earthing is no longer common practice.

Modern practice is to use nominally insulated track. For example the UK Railway Group Standard GL/RT1254 (Railtrack, 2000) says that DC systems "shall be designed, installed and maintained so as to ensure that there are no deliberate low resistance points of contact between the traction return circuit and the general mass of the earth". All current International, European and National Standards follow the joint principles of a nominally insulated track, and a low along-track resistance. And new railways might be compelled by their planning consent to comply with some numerical specification. For example, the European Standard EN 50122-2:1998 (European Standard, 1998) recommends maintaining a track-to-ground conductance of not more that 0.5 S/km for single track on surface track, and not more than 0.1 S/km in tunnel; however, particularly in urban areas, it is now common to specify the smaller figure (or less) for all track. (Note that this figure of 0.1 S/km is a *distributed linear* conductance; the actual total conductance in siemens between the rail and the body of the earth is this figure multiplied by the length of track in km. Some texts use the equivalent linear resistance value of 10 Ω.km.)

Existing railways will lie somewhere between these extremes, with the majority having a nominally insulated track; if you are interested in a particular railway you need to consult its power-supplies engineer to obtain the relevant design information (though the actual situation might be a lot poorer than the design specified).

The aim of reducing earth leakage is not out of consideration for geomagneticians, but because of the need to reduce electrolytic corrosion of the infrastructure of other utilities. The problem is that if there are DC currents flowing in the ground, these will tend to be collected and concentrated into any metal pipelines, metal-armouring of cables, etc. that happen to be around. Nothing much happens where the (positive) current enters the metal, but where it leaves the metal there is electrolytic corrosion. A rough figure is that 1 A flowing for one year results in a loss of 10 kg of iron.

Note that the situation is quite different for AC railway electrification. The steel rail has a much higher impedance to 50/60 Hz AC current than to DC current, so for safety the track *is* connected to earth at frequent intervals. Other measures are taken to return the traction current to the substation, and to reduce the earth leakage current.

When we are looking at how the traction return current can leak into the ground, we are concerned only with the voltage to earth of the running rails. Fortunately, the discussion in Section 2 above means that we can ignore all the details of the circuitry *above* the rails. For a train at a *given* position we simply connect the appropriate constant current generator between the track at that position and the track at the position of each substation. To a good approximation, the current in each generator is independent of what other trains are on the track, and to how much current is leaking to the ground. This approximation makes the following discussion and algebra *much* simpler.

3.3 Qualitative discussion of diffuse earth-leakage currents

To start with, consider an isolated length of uniform track, nominally insulated, but having a small uniform electrical conductance with the ground. There is a substation at one end and a train at the other (ignore the fact that the train is moving)—see Fig. 3(a). The flow of the traction current along the running rails back to the substation creates a potential gradient along the rail (of the order of 20 V/km in urban situations). To the approximation that the track is uniform, and that the total leakage current is much smaller than the traction current, this potential gradient will be constant between the substation and the train. Because of the weak continuous electrical contact with the ground, current will leak into the ground where the rail is positive with respect to the ground, and return to the rail where it is negative. An equilibrium will be reached, in which as much leakage current enters the rail as leaves it, such that near the train the rails will be positive with respect to the ground, and near the substation they will be negative, with a 'virtual earth' at some intermediate point. Therefore there will be a leakage current density from the rail into the ground near the train, decreasing to zero, and then becoming an increasing current density from the ground into the rail, nearer the substation.

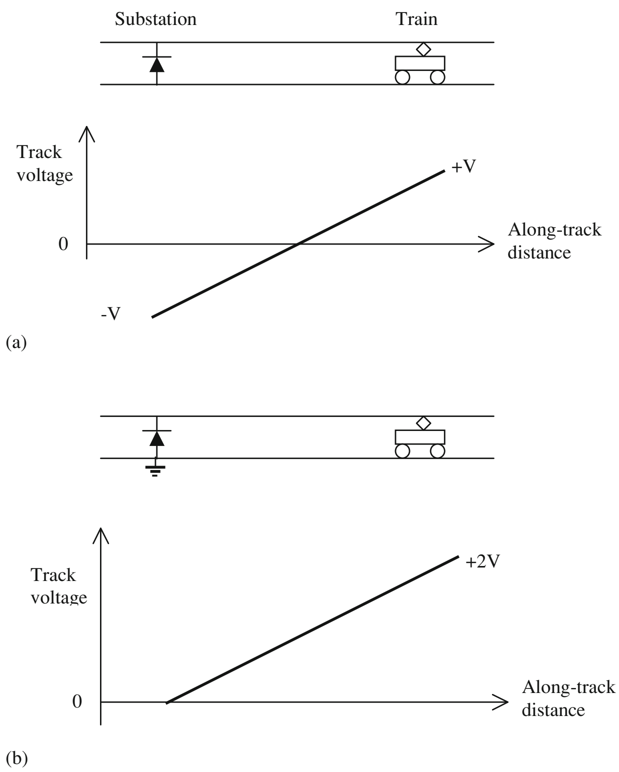

Fig. 3. For small earth leakage the along-track current, and hence the along-track potential gradient, is approximately constant between the substation and train. The local earth-leakage current is proportional to the track-ground voltage. The resulting track voltage is shown (a) when the track is floating, and (b) when the track is earthed at the substation. In (b) all the leakage current returns to the substation through its earth.

If we now extend the track for some distance in both directions, then 'beyond' the train (in the direction away from the substation) the track will be at a roughly constant positive potential, so the current leakage into the ground will continue beyond the train. Similarly, the return of current from the ground to the track will continue beyond the substation. A new equilibrium will be reached, and the position of the virtual earth will now depend also on the lengths of the two track extensions.

If we ignore small local surface electrochemical effects (of the order of a volt) at the rail/ground interface, the rail/ground resistance network is linear. So, if there is more than one substation, and/or more than one train, once the current division between substations has been calculated for each train, one can calculate the voltage distribution, and hence leakage-current, distribution, appropriate for *each* partial traction current *separately*, and then simply add the leakage current distributions to obtain the overall distribution in the more complicated situation.

3.4 Point leaks to earth

There will *always* be diffuse leakage of the sort discussed above. However there might *also* be deliberate or accidental 'point' leaks, places where there is a local low-resistance electrical connection from the track to the ground. For example, in some railways it was the practice to earth (i.e. connect to a large-area, low-resistance, electrode in the ground) the negative busbar (connected to the running rails) at the substation. If the substation earth has a very much lower resistance to ground than the integrated effect of the diffuse rail-ground conductance, this has the effect of 'clamping' the track potential at the substation to zero potential. Consider the situation of Fig. 3(a) where there is only one substation. For small diffuse leakage, the current along the track (and hence the potential *gradient*) is essentially constant at the full traction current. If we now earth the substation, the whole potential graph has to move bodily upwards, and the track potential is now above zero everywhere—Fig. 3(b). About twice as much track is now positive than was before, and the average positive voltage is also doubled, so the amount of current leaking diffusely into the ground will be roughly quadrupled. If all the substations have their negative busbars earthed, the situation is essentially the same, with again a quadrupling of any diffuse leakage.

Another common source of a (roughly) point leak, is the rolling-stock depot, where the sidings give a long length of track in a small area, and for safety in the workshop areas the track there has to be near earth potential. The depot then acts as a permanent low-resistance earth to the whole of the railway system. To avoid this, modern practice is to isolate the depot electrically from the rest of the system.

But the main problem caused by earthing at the depot or substations is that it very much aggravates earth-leakage problems throughout the whole system. Suppose that there were another, possibly accidental, low-resistance point leak, call it leak A. If this were the only point leak, any current through it has to return to the track diffusely, so is limited. But if there is second low-resistance earth connection at a more negative track position, for example at a depot, the current through leak A can be much increased.

An extreme example occurred when the Tyne and Wear Metro (UK) was opened in 1980 (Lowes, 1987a, b). At the time, the relevant section of line was fed by a single substation, next to an earthy depot. At the end of the line, 4 km away, was a new underground station, with the tunnel lined by cast-iron segments. Because of various mechanical failures, there was a low-resistance connection between the track in the tunnel, and the tunnel lining. As a result, when a train drew away from the station, of the 1000 A current, 400 A returned to the substation through the ground rather than through the rails! Another example occurred when an extension was built, with the track connected through, but having a separate overhead system; at one substation a temporary safety earth on the negative busbar was accidentally left in place when train running commenced. For a train 2 km nearer the depot than the substation, of the 1000 A current, 40 A travelled 10 km further along the track to the depot, returning underground, partly via a convenient gas main!

One problem for the railway engineer is that he has to find some compromise between reducing earth-leakage, but also not allowing any part of the track, and hence train metalwork, to reach a dangerous voltage with respect to earthed station metalwork. In some systems a conducting grid is built-in under the track, so as to collect any leakage current and return it to the substation, while other systems connect the substation negative busbar to earth through a diode. Older systems might have spark gaps, to connect metalwork to the rails in fault conditions, but these remain conducting after the fault is cleared; modern semi-conductor equivalents revert to high resistance once the fault has cleared. Some systems might have 'intelligent' devices which apply a temporary earth to the rail (for example at substations or stations) only when the touch potential would otherwise be too high; in such systems there could be additional point leak currents into ground, at fixed locations but erratic in time.

4. Quantitative Discussion of Diffuse Leakage Currents

I first use an approximate calculation for a simple situation, to illustrate the behaviour of the system. Then I give an exact calculation, and show how the approximation is related to it.

4.1 Approximate calculation

Consider the simple case of Fig. 3(a), where there is one substation, one train, and only diffuse leakage. I will use the symbol ρ for the along-track rail resistance per unit length, and σ for the track-ground conductance per unit length. As a first approximation, assume that the total leakage current is small compared with the traction current, so that the current in the rail is almost constant at I_T (positive from train to substation), with the potential gradient along the track being constant at $I_T\,\rho$. The train is distance b from the substation, and the total track length is L. If the track voltage with respect to ground is $-V_{ss}$ at the substation, then the voltage at distance x from the substation is

$$V(x) = -V_{ss} + I_T\,\rho x, \tag{1}$$

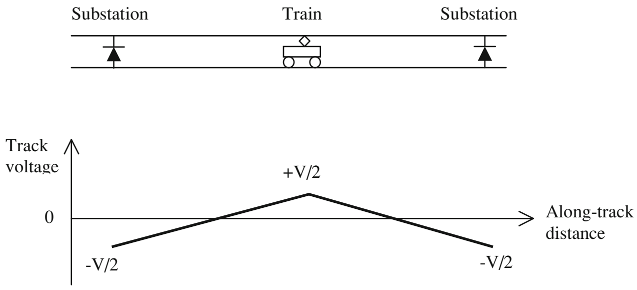

Fig. 4. Track voltage for floating track for a train midway between two substations.

and constant at $V(b)$ for $x > b$ (beyond the train). The leakage current density (current per unit length, positive when from track to ground) at any point will be $\sigma V(x)$. To ensure conservation of traction current, the integral of this leakage current over the whole track must be zero, and it is this condition that determines $-V_{ss}$, which is a function of the train position b. We are often concerned only with the *total* current leaving the positive (or entering the negative) part of the track; this is σ times the area under the positive part of the $V(x)/x$ graph, and it is easy to show that it is a maximum when the train is at the end of the track, $b = L$, giving the voltage distribution

$$V(x) = \rho I_T \left(-\frac{1}{2}L + x\right). \tag{2}$$

The local leakage current is $\sigma V(x)$, and the total leakage current, given by integrating over half the voltage curve is

$$I_L = (1/8)\left(\sigma\rho L^2\right) I_T = (\sigma L/4)\, V(L), \tag{3}$$

where $V(L) = \frac{1}{2}\rho I_T L$ is the track voltage at the train. As seen from the train, we can think of the track as having an effective leakage input conductance of $\sigma L/4$; the factor of 1/4 arises because only half the track is positive, and the average positive voltage is $\frac{1}{2}V(L)$.

However if the rail potential is clamped to earth at the substation, as in Fig. 3(b), the voltage distribution between the substation and the train is

$$V(x) = \rho I_T\, x. \tag{4}$$

Again the leakage current is a maximum when the train is at the end, $b = L$, and the total leakage current is then

$$I_L = (1/2)\left(\sigma\rho L^2\right) I_T, \tag{5}$$

four times larger than if the track were floating. (For a finite-resistance substation earth, the virtual earth moves away from the substation, so the increase in leakage caused by the substation earth will not be as large.)

If this isolated length of track has an identical substation at each end, it can be shown that the maximum total leakage current now occurs when the train is midway between the substations (Fig. 4), with $\frac{1}{2}I_T$ flowing to each. This maximum total leakage current is now

$$I_L = (1/32)\left(\sigma\rho L^2\right) I_T, \tag{6}$$

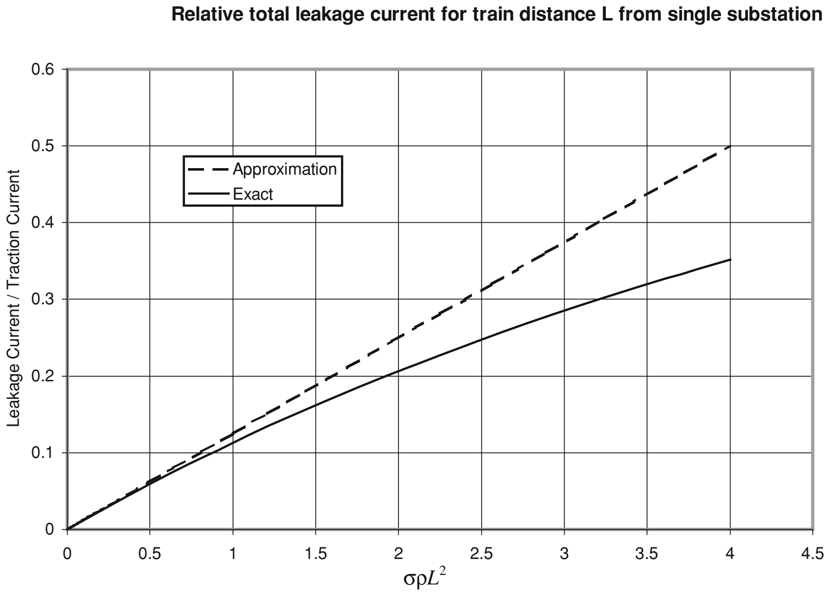

Fig. 5. For a train at the end of a track of length L from the substation, a plot of the ratio I_L/I_T (total leakage current/traction current) against the non-dimensional parameter $\sigma\rho L^2$, where σ is the track-to-earth leakage (linear) conductance, and ρ is the along-track (linear) resistance. The full curve is the exact solution of Eq. (15), and the dashed curve is the linear approximation of (3).

a factor of four reduction over the case with a single substation. As above, if the track is earthed at the two substations, the track-voltage curve is moved upwards, and the total leakage current is again increased by a factor of four.

We will see below that $\sigma\rho L^2$ is an appropriate non-dimensional scaling factor, and that the approximation (3) used above is valid for small values of this factor. Figure 5 plots both the approximation and the exact value of I_L/I_T against $\sigma\rho L^2$. Putting $\sigma = 0.1$ S/km, $\rho = 0.02$ Ω/km, $L = 4$ km, giving $\sigma\rho L^2 = 0.032$, we see that in this example the maximum I_L is only $0.004 I_T$, so this approximate approach is justified for this high-quality track. For poorer track (larger values of the scaling factor), the approximation overestimates the leakage, but the error is only about 10% for $\sigma\rho L^2 = 1$, when I_L/I_T is about 0.11. This suggests that the approximation will be good even in more complicated situations, provided the leaked current is not more than about $0.1 I_T$. For $\sigma\rho L^2 > 1$ the approximation worsens rapidly—the ratio asymptotes to 1, while the approximation increases without limit—but is still good to better than 50% at $\sigma\rho L^2 = 4$.

4.2 Exact analysis for one segment of track

Consider any one segment of uniform track between a train and a substation. In the absence of leakage, the along-track current $I(x)$ at distance x from the substation would be independent of x. However if there is significant leakage, $I(x)$ will vary with x. If there is a leakage current density $j_L = \sigma V(x)$, conservation of current gives

$$\mathrm{d}I(x)/\mathrm{d}x = j_L = \sigma V(x). \tag{7}$$

Applying Ohm's Law to the current $I(x)$, we get

$$\mathrm{d}V(x)/\mathrm{d}x = \rho I(x). \tag{8}$$

Combining these two equations gives

$$\mathrm{d}^2 I(x)/\mathrm{d}x^2 = \sigma\rho I(x). \tag{9}$$

This has the general solution

$$I(x) = Ae^{\alpha x} + Be^{-\alpha x}, \tag{10}$$

where A and B are parameters which have to be chosen to fit the boundary conditions at the ends of the segment, and $\alpha = (\sigma\rho)^{0.5}$. If there is more than one segment, there is a similar equation for each segment, with appropriate boundary conditions at each junction; this algebra is presented by, e.g., Yanagihara (1977) and Lee and Wang (2001). For the example used above ($\sigma = 0.1$ S/km, $\rho = 0.02$ Ω/km) α is about (1/22) km^{-1}.

For the simple case of Fig. 3(a), with the train at the end $x = L$, and the substation at $x = 0$, the two boundary conditions are that $I(L) = I(0) = I_T$, with the full traction current leaving the train and entering the substation. The solution is then

$$I(x) = Ae^{\alpha x} + Be^{-\alpha x} \tag{11}$$

with

$$\begin{aligned} A &= I_T\left(1 - e^{-\alpha L}\right) / \left(e^{\alpha L} - e^{-\alpha L}\right), \\ B &= I_T\left(e^{\alpha L} - 1\right) / \left(e^{\alpha L} - e^{-\alpha L}\right), \\ \alpha &= (\sigma\rho)^{0.5} \end{aligned} \tag{12}$$

giving

$$I(x) = I_T\left(e^{\alpha x} + e^{\alpha(L-x)}\right)\left(1 - e^{-\alpha L}\right) / \left(e^{\alpha L} - e^{-\alpha L}\right). \tag{13}$$

Thus the along-track current is symmetrical, with a minimum at the centre point—see Fig. 6. The corresponding track voltage is

$$\begin{aligned} V(x) &= (1/\sigma)\,\mathrm{d}I(x)/\mathrm{d}x = (\rho/\alpha)\left(Ae^{\alpha x} - Be^{-\alpha x}\right) \\ &= I_T(\rho/\alpha)\left(e^{\alpha x} - e^{\alpha(L-x)}\right)\left(1 - e^{-\alpha L}\right) / \left(e^{\alpha L} - e^{-\alpha L}\right); \end{aligned} \tag{14}$$

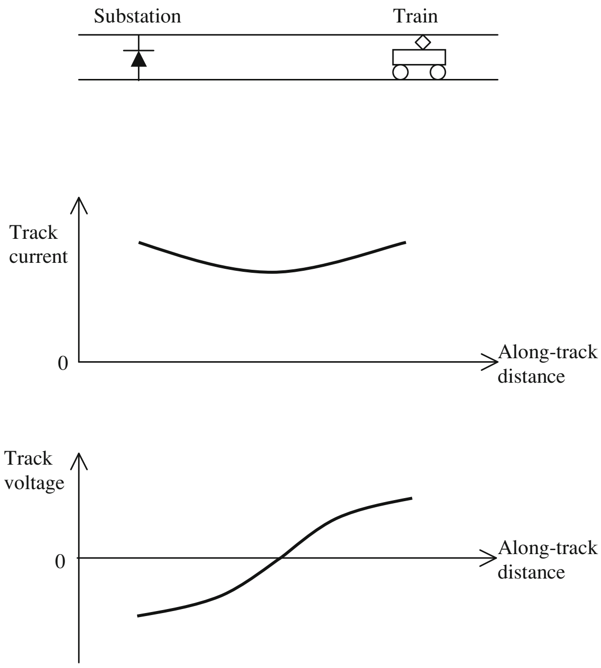

Fig. 6. Track current and voltage for floating track with significant earth-leakage. The local earth-leakage current density is proportional to the track voltage.

note that this solution automatically ensures that the absolute voltage level is such that as much leakage current leaves the rail as enters it. The leakage current density is $j_L = \sigma V(x)$, and the total leakage current is

$$\begin{aligned} I_L &= I(0) - I(L/2) = I_T\left[1 - 2\big/\left(e^{-\alpha L/2} + e^{\alpha L/2}\right)\right] \\ &= I_T\left(e^{-\alpha L/2} + e^{\alpha L/2} - 2\right)\big/\left(e^{-\alpha L/2} + e^{\alpha L/2}\right). \quad (15) \end{aligned}$$

The value of I_L/I_T is plotted as a function of $(\alpha L)^2 = \sigma\rho L^2$ in Fig. 5, together with the approximation $(1/8)\sigma\rho L^2$ derived above. This figure can be used to determine the leakage for other combinations of parameters, and also for the case when the train is half-way between two substations—see below.

4.3 Quadratic and linear approximations

The exponential functions in (13) and (14) can be expanded as power series in α; for $I(x)$ only the even powers occur, and for $V(x)$ only the odd powers. If αx is small (13) and (14) can be roughly approximated by linear functions, and well approximated by quadratic functions. Truncating both $I(x)$ and $V(x)$ at the quadratic terms gives the approximations

$$I(x) = I_T\left[1 - \frac{1}{2}\alpha^2 x(L-x)\right], \quad (16)$$

and

$$V(x) = I_T(\rho/\alpha)\alpha\left(x - \frac{1}{2}L\right) = \rho I_T\left(x - \frac{1}{2}L\right), \quad (17)$$

with

$$j_L(x) = \sigma V(x) = \sigma\rho I_T\left(x - \frac{1}{2}L\right) = \alpha^2 I_T\left(x - \frac{1}{2}L\right). \quad (18)$$

In this approximation the along-track current $I(x)$ is parabolic, with a minimum at $x = \frac{1}{2}L$, and the potential gradient is constant, at the value corresponding to the full traction current. The total leakage current I_L is given either by taking the area under either half of the voltage curve, or in this quadratic approximation by using $I_L = I(0) - I(L/2)$, and is

$$I_L = (1/8)\sigma\rho L^2 I_T = \sigma(L/4)(\rho L I_T/2) = (\rho L/4)\,V(L), \quad (19)$$

as in (3).

Yanagihara (1977) used this quadratic current approximation separately for each track segment, assuming values for σ and ρ, and using the parameters that gave the correct theoretical rail current at the segment boundaries and an intermediate point.

If the expansions of the exponential parts of $I(x)$ and $V(x)$ are further truncated at the terms linear in α, we get $I(x) = I_T$ and $V(x) = \rho I_T\left(x - \frac{1}{2}L\right)$, the approximations used above in Section 4.1. Using the area under the voltage curve gives the same total leakage current as the quadratic approximation.

Georgescu *et al.* (2002) used a different approximation, that the leakage current density $j_L(x)$ was *constant* (independent of position), but did not explain how this was justified. However this corresponds to the whole of the relevant length of track being at constant potential with respect to earth, a completely unrealistic situation. Unfortunately Pirjola *et al.* (2007), in their detailed calculations of the magnetic fields produced by the leakage currents, followed the Georgescu *et al.* model.

4.4 Exact analysis for two segments of track

If as in Fig. 4 there are substations at each end of the track, the two segments of track each have their own solution. If the train is at $x = b$ we have

$$I_1(x) = Ae^{\alpha x} + Be^{-\alpha x} \quad \text{for } 0 < x < b \quad (20)$$

and

$$I_2(x) = Ce^{\alpha x} + De^{-\alpha x} \quad \text{for } b < x < L. \quad (21)$$

If the train is taking currents $f_1(b)I_T$, $f_2(b)I_T$ from the left and right substations, the boundary conditions are

$$\begin{aligned} I_1(0) &= f_1(b)I_T, \quad I_1(b) - I_2(b) = I_T, \\ I_2(L) &= -f_2(b)I_T, \quad V_1(b) = V_2(b). \end{aligned} \quad (22)$$

As in the discussion of Section 4.1 above, we can expect the maximum leakage to occur when the train is at the midpoint $b = L/2$. The current distribution in the one segment is then simply the mirror image of that in the other segment, and the total leakage current in each segment is given by Eq. (15) and Fig. 5, except that I_T is replaced by $I_T/2$ and L is replaced by $L/2$. As noted below (6), in the linear approximation adding the second substation reduces the total leakage current by a factor of four; in the exact solution the reduction is somewhat less.

4.5 The effect of track extensions

In the above discussions I ignored the existence of any track outside the region $0 < x < L$, but in practice there will almost always be such extensions. One complication is that distant substations will now be providing a small fraction of the traction current of the train being considered; if necessary these currents could be determined, and considered separately in the magnetic field calculations. However in the present context a more serious effect is that these extensions can significantly increase the leakage current from a given train.

In the small-leakage approximation of Section 4.1, we can take the voltage along the extension as constant. Then in the one train, one substation, situation of Fig. 3(a), if the train is 4 km from the substation and the track is extended by $l = 4$ km at each end, the voltage distribution between the substation and train is unchanged, and the extensions are at $V = \pm\frac{1}{2}\rho I_T L$, adding $\frac{1}{2}\sigma\rho I_T L l$ to the leakage current, four times that given by the central section. Again this approximation breaks down for larger values of $\sigma\rho l^2$; while the approximation predicts I_L increasing without limit as l increases, in practice the leakage current has to flow along the track, and its own ohmic voltage drop limits the total leakage. The formal solution is as follows.

If the track voltage at the train at $x = L$ is V_L, for an extension of length l the track current at the end $x = L + l$ is zero. With these boundary conditions the track current in the extension is

$$I_3(x) = (\sigma/\alpha)\, V_L \left[e^{-\alpha(x-L)} - e^{-2\alpha l} e^{\alpha(x-L)}\right] / \left(1 + e^{-2\alpha l}\right)$$
$$= (\sigma/\alpha) V_L \left[e^{-\alpha(x-L)} - e^{-\alpha(L+2l-x)}\right] / \left(1 + e^{-2\alpha l}\right); \quad (23)$$

we can think of the input voltage V_L being 'reflected' at the end of the extension, giving a negative image at $2l$ from the train. As seen from the train at $x = L$, the track extension has a conductance to earth of

$$S_E = I_3(L)/V_L = (\sigma/\alpha)\left(1 - e^{-2\alpha l}\right) / \left(1 + e^{-2\alpha l}\right). \quad (24)$$

For small αl the extension conductance S_E is σl as above. For large αl the conductance S_E asymptotes to σ/α, so that the extension behaves like a length $1/\alpha$ of track held at constant voltage. For the numerical example used above $1/\alpha$ is 22 km; in this situation a long track extension can apparently give five times more leakage than the track between the substation and the train. However any significant leakage in an extension will itself change the boundary conditions, and hence the voltage distribution in the central section. It is also unlikely that there will be such a long track extension. I now discuss qualitatively the various situations and give limits for the maximum likely increase in leakage.

If the train is between two substations and the track is earthed at the substations, any track beyond the substations has no effect.

If the train is between two substations, and the track is floating, as in Fig. 4, then beyond the substations the track is at some negative voltage. Leakage into these regions must be balanced by increased leakage from the positive region near the train, so in the central region the whole voltage graph must move more positive, by the track voltages at the substations becoming less negative. The increase in overall leakage current will depend on how long and leaky the extensions are. But the *maximum* (and unlikely) increase of leakage current is when the voltages at the substations have been dragged up to earth potential, and as we saw above this increases the leakage current by a factor of four.

The more difficult situation is for a train beyond the last substation, as in Fig. 3. Any track beyond the substation (away from the train) will have the same effect as discussed in the previous paragraph. Formally, as discussed above, any extension beyond the train could add substantially to the leakage current; the longer the extension the more leakage there will be. However the length of the extension is unlikely to be more than about half the typical distance between substations, so the *actual* extra leakage will usually be quite limited. (Even if the track continues as a non-electrified railway, there should be insulated rail joints at the boundary.) And in any case the maximum *total* leakage, from *all* the track, will be when the train is itself at the end of the track, and for most observation positions this will also be the train position giving maximum disturbance!

The full solution for track voltage and along-track current, for a system having a train at an arbitrary point between two substations, and with an arbitrary extension at each end (characterised by their input resistances $1/S_E$), is given by Lee and Wang (2001), both for floating track, and for track earthed through a finite resistance at the substations.

5. The Magnetic Field Produced by the Railway System

Each segment of track between a train and a substation can be treated separately, and the resultant magnetic fields added vectorially; in what follows I_T is the traction current appropriate to one particular segment. For a busy railway, at any one time there might be more than one train visible magnetically.

For simplicity I will discuss only 'positive' traction current; any regenerated current will usually be in the reverse direction. For simplicity in description, I will assume that the railway is everywhere on a horizontal ground surface; except in hilly country any errors will usually be small compared with the effect of the uncertainty in the earth-leakage distribution.

Magnetic fields are produced by the large traction currents flowing above ground, but these fields reduce rapidly with distance. The fields produced by the much smaller leakage currents reduce more slowly with distance, and will probably dominate at geomagnetically relevant distances.

The actual rail current will in places be less than the traction current—see Fig. 7(a). The magnetic-field calculation is simplified if we consider the overall current distribution as the sum of two separate circuits—Fig. 7(b); this separation is possible as the magnetic field is a linear function of current. The first current circuit is entirely above ground, and consists of a succession of 'line' currents all carrying the full traction current I_T. This current flows vertically up through the substation, horizontally along the overhead wire, vertically down through the train, and returns horizontally through the running-rails just above the ground surface. (The difference between I_T and the actual rail current is given by the second circuit.)

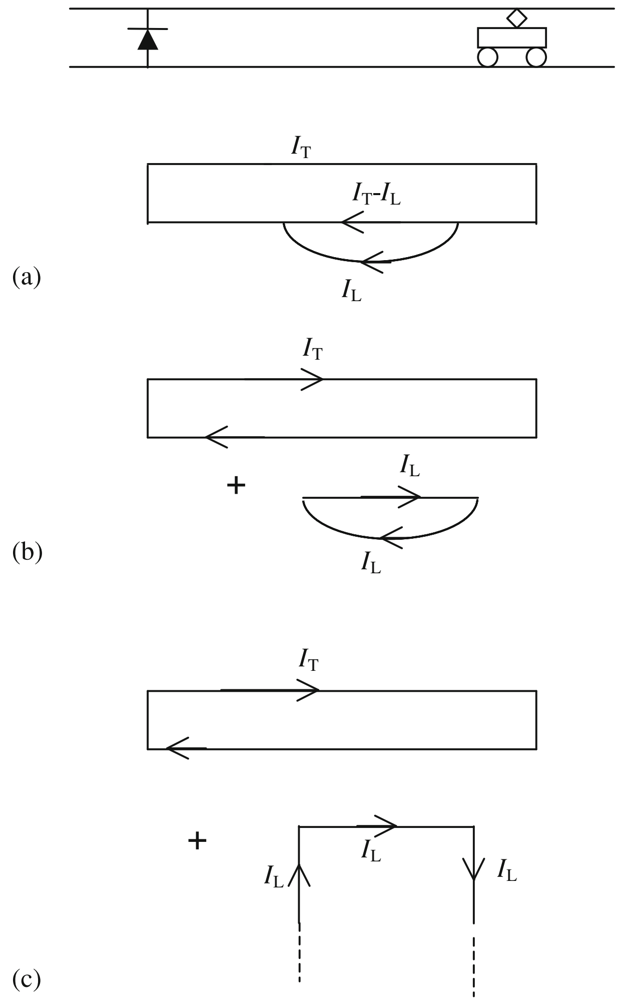

Fig. 7. (a) Schematic diagram showing that some of the traction current I_T returning through the rails leaks to and from the ground. (b) The actual current system is equivalent to the sum of two separate circuits: (i) the full traction current flowing along the overhead and rails, and (ii) a leakage current circuit having the leakage current flowing diffusely through the ground and returning along the rails. (c) From the magnetic point of view, the diffuse leakage current is replaced by two vertical semi-infinite line currents.

The second current circuit is that of the earth-leakage currents. At the ground surface are the line of sources (sinks) of the rail-ground leakage current density j_L being injected into (extracted from) the ground. Just above the surface these are connected by the appropriate integrated along-track leakage current along the rail; this is in the opposite direction to the traction current, and is what reduces the net along-track current to its actual value. Under the surface these currents return from the sources to the sinks in a diffuse 3-dimensional flow. However we will see below that in terms of its magnetic effect the diffuse flow between any one source/sink pair can usually be replaced by the sum of two semi-infinite vertical lines of current—Fig. 7(c).

The magnetic effects of these two current circuits will be discussed separately, in Sections 5.2 and 5.3. In each case the circuit is divided into separate simple elements, and the magnetic field is calculated for each element and then needs to be summed. So it is useful to first recall some basic results.

5.1 The magnetic field produced by some simple circuit elements

I will assume that the whole region has vacuum permeability μ_0. The contribution to the flux density $\boldsymbol{B}$ (unit T, and which from now on I will call simply (magnetic) field) produced at an observation point vector $\boldsymbol{r}$ from a short straight line current element $I\mathrm{d}\boldsymbol{s}$ is then given by Biot-Savart's Law as

$$\mathrm{d}\boldsymbol{B} = \mu_0 I(\mathrm{d}\boldsymbol{s} \times \boldsymbol{r}/r)/4\pi r^2 = \mu_0 I(\mathrm{d}\boldsymbol{s} \times \boldsymbol{e}_r)/4\pi r^2. \quad (25)$$

Although intended to apply to an infinitesimal length $\mathrm{d}\boldsymbol{s}$, at distant points this expression is quite a good approximation for a finite straight-line length L of current. For example, putting $\mathrm{d}s = L$ in (25) gives results good to about 10% for an observation point at distance (perpendicular to the current) $R > L$, and to 50% for $R > \frac{1}{2}L$. For more accuracy, and/or nearer points, (25) has to be integrated over the relevant part of the circuit:

$$\boldsymbol{B} = \mu_0 \int I(\mathrm{d}\boldsymbol{s} \times \boldsymbol{e}_r)/4\pi r^2. \quad (26)$$

For a finite-length straight-line element of current I (Fig. 8), the field at the observation point P is clockwise circular when looking in the direction of the current, and has magnitude

$$B = \mu_0 I\,(\cos\theta_1 - \cos\theta_2)\,/4\pi R, \quad (27)$$

where R is the perpendicular distance from the observation point to the current line, and θ_1 and θ_2 are the angles between the line and the vectors from the ends of the line to the observation point. If end 1 is at infinity, giving $\theta_1 = 0$,

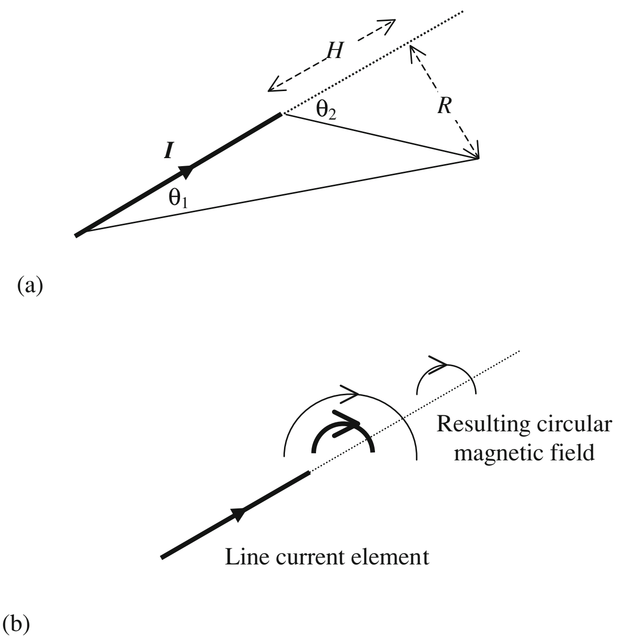

Fig. 8. (a) Geometry for calculating the magnetic field given by a finite-length straight-line current. (b) The resulting circular magnetic field is shown for the half-space above the current element; the field magnitude reduces with (cylindrical) radial distance R from the current line, and (more slowly) with distance H beyond the end of the current element.

we have

$$B_{\infty\theta} = \mu_0 I \, (1 - \cos\theta_2) \, /4\pi R, \tag{28}$$

and if the observation point is level with end 2, so that $\theta_2 = \pi/2$, this becomes

$$B_{\infty/2} = \mu_0 I/4\pi R. \tag{29}$$

For an infinite-length line current ($\theta_1 = 0, \theta_2 = \pi$) we have the Ampere-Law result

$$B_\infty = \mu_0 I/2\pi R. \tag{30}$$

If one is close enough to the current, and not near its ends, (30) can also be used approximately for finite-length current elements. For example, for a current of length L, for positions within the middle half of the element, (30) is accurate to about 10% out to $R = \frac{1}{4}L$, and to 50% out to $R = \frac{1}{2}L$.

Another useful result is that for the magnetic field given by two parallel line currents, spaced h apart and carrying equal currents I in opposite directions, e.g. railway overhead and rail. It is now convenient to use cylindrical polar co-ordinates (R, λ, z), with $\lambda = 0$ in the plane perpendicular to the plane of the currents; in the railway case z is the distance x along the track. The resultant field is everywhere still in planes perpendicular to the currents. Near the wires we have to calculate the magnetic fields of each current separately and add them vectorially, and this is what Pirjola *et al.* (2007) did for all observation points. But for distances R large compared with h (say $R > 20$ m for a railway), we have the field of a 'line-dipole', having dipole moment Ih per unit length oriented perpendicular to the plane of the current separation. It is easily shown that for an infinite length line-dipole, the field components are

$$\begin{aligned} B_{r\infty} &= \left(\mu_0 I h/2\pi R^2\right) \cos\lambda, \\ B_{\lambda\infty} &= \left(\mu_0 I h/2\pi R^2\right) \sin\lambda. \end{aligned} \tag{31}$$

(Note that unlike the case of the 3-dimensional point dipole, there is no factor of two difference between the two terms.) In cartesian co-ordinates, with (x, y) in the $\lambda = (0, 90^\circ)$ directions, the components are

$$\begin{aligned} B_{x\infty} &= \left(\mu_0 I h/2\pi R^2\right) \cos 2\lambda, \\ B_{y\infty} &= \left(\mu_0 I h/2\pi R^2\right) \sin 2\lambda. \end{aligned} \tag{32}$$

See Fig. 9 for an illustration of this line-dipole field, and the co-ordinate system.

As above, for an observation point level with one end of a semi-infinite line dipole we have

$$\begin{aligned} B_{r\infty/2} &= \left(\mu_0 I h/4\pi R^2\right) \cos\lambda, \\ B_{\lambda\infty/2} &= \left(\mu_0 I h/4\pi R^2\right) \sin\lambda, \end{aligned} \tag{33}$$

and for the general case of a finite length of line-dipole we have

$$\begin{aligned} B_r &= \left(\mu_0 I h/4\pi R^2\right) \left(\cos\theta_1 - \cos\theta_2\right) \cos\lambda, \\ B_\lambda &= \left(\mu_0 I h/4\pi R^2\right) \left(\cos\theta_1 - \cos\theta_2\right) \sin\lambda, \end{aligned} \tag{34}$$

where θ_1 and θ_2 are again as in Fig. 8. Just as (30) for an infinite line-current can be used to give a good approximation not too far from a finite-length line-current, so also can (31) be used not too far from a finite-length line-dipole current; provided $R \gg h$ the relative approximation is the same in the two cases.

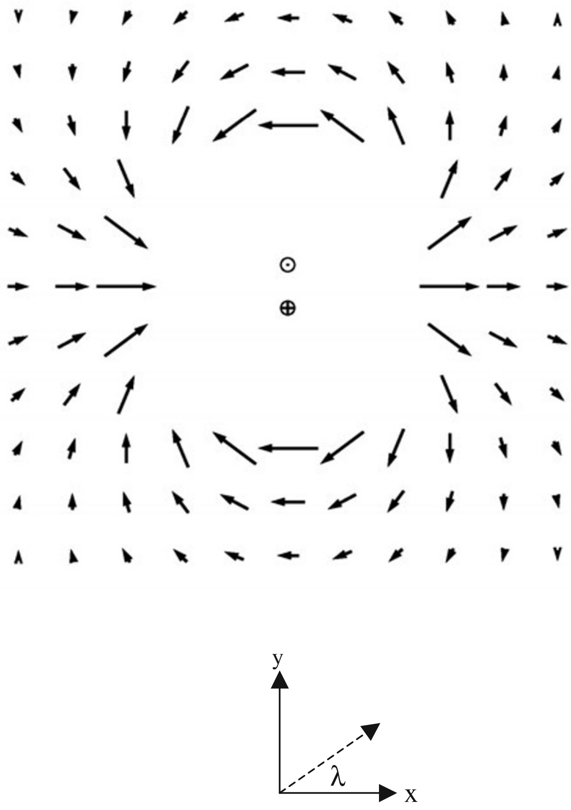

Fig. 9. A cross-section of the line-dipole field given by two anti-parallel currents.

A third useful result is for the field produced by a current injected at a point on the surface of a semi-infinite uniform ground, and spreading radially to infinity. Clearly the field must have axial symmetry, and it is easy to show that it must everywhere be purely circumferential. Dupouy (1950), Yanagihara (1977), and Tokumoto and Tsunomura (1984) used the result (without proof) that for a uniform ground having a plane horizontal top surface the magnetic field produced at any point immediately above the surface is the same as that produced by a semi-infinite vertical line current starting at the injection point (Eq. (29) above):

(field given by current I injected at surface)
= (field given by semi-infinite line current I). (35)

Pirjola *et al.* (2007) pointed out that this result is equivalent to the 'Fukushima Theorem' (Fukushima, 1976) used in ionospheric situations. The proof of Fukushima is for a thin sheet of current, but Pirjola *et al.* pointed out that it could be generalised to a ground having conductivity varying only with depth. (Previous authors, such as Tokumoto and Tsunomura (1984), had assumed that if the conductivity changed with depth this would change the above-surface field.) In fact the same approach, integrating $\boldsymbol{B}$ round a horizontal circle in the non-conducting region, shows that the result is applicable *anywhere* above the flat ground surface.

5.2 The magnetic field produced by the traction-current circuit

This circuit consists of two very short vertical sections (through the substation and train), and two long horizontal sections (the overhead and track), having a vertical separation h of about 5 m. Although Pirjola *et al.* (2007) included the vertical sections in their calculations, at realistic distances the field they produce is trivial compared with the other contributions, being less than 1 nT at 100 m from 1000 A flowing over 5 m, and falling-off with distance R as $1/R^2$; I will ignore it.

That leaves the two roughly horizontal (overhead and rail) line currents. Georgescu *et al.* (2002) and Pirjola *et al.* (2007) calculated the fields separately (Eq. (27) above) for the two currents, and then subtracted them. However using the line-dipole expression (34) gives less than 10% error at distances greater than about 10 m, and less than 1% by 25 m, so I will use the line-dipole approach, which greatly simplifies the algebra; this approach was used by Dupouy (1950). For horizontal track there is a dipole-type field in vertical planes—see Fig. 9; the angle λ of Eqs. (31–34) is measured from the horizontal plane midway between the rails and overhead wire. (Pirjola *et al.* (2007) measured from the plane of the rails, but the difference is trivial at realistic distances.) For measurements level with the track, the field produced is horizontal and perpendicular to the line of the track; it is directed toward the track when the overhead current is from left to right. For observation points above or below the track level, the horizontal component falls off as $\cos 2\lambda$; there is also a vertical component, of the same amplitude, proportional to $\sin 2\lambda$. For distances small compared with the track length, we can use the infinite-length approximation (31–32), and the field falls-off as $1/R^2$, where R is the perpendicular distance from the track. But in practice, as R is increased this contribution soon becomes smaller than that produced by the along-track leakage current, and so can usually be ignored; see the next section.

(Although only of academic interest in the present context, at larger distances the finite length of track becomes important; the finite-length (34) should be used, and the variation with distance will become more rapid. At distances large compared with the length L of the railway segment, the field can be approximated as that of a point (3-dimensional) dipole of moment $I_T L h$, with the field decreasing as $1/R^3$.)

5.3 The magnetic field produced by the earth-leakage currents

5.3.1 The above-ground current The above-ground part of this circuit is an along-track line current, varying with position as current is collected from, and lost back to, the ground. The resulting field is circular about the track, being strongest where the along-rail current is largest. For observation points near the horizontal this current will give a vertical field, downward for a current flowing from left to right. Close to the track use the infinite-length expression (30); this is what Linington (1974) and Lowes (1987a, b) did. This field falls off only as $1/R$, and at some distance will become larger than the $1/R^2$ line-dipole field from the traction current. (For example, at 100 m the horizontal line-dipole field given by a traction current of 1000 A, and the vertical circular field given by 50 A leakage current, are both 100 nT.) Close to the track this vertical field will also be larger than the horizontal field coming from the underground part of the leakage-current circuit, though the two fields will become of the same magnitude at distances comparable with the track length—see below. If further away, use the finite-length (27); the fall-off with distance will then approach $1/R^2$.

A point leak will give a current that is constant along the relevant part of the track. But diffuse leakage will give a current that varies along the track. If the variation is known, for example the exponential variation of Section 4.2, then the appropriate integration can be made (algebraically or numerically) for each straight-line section of track. However in many cases this variation will not be known in detail, and then a simpler approximate calculation, based on an estimate of total leakage current, would be appropriate.

5.3.2 The underground current If the half-space by which we represent the ground has uniform conductivity, the diffuse current flow in the ground between a particular injection point (source) and a particular extraction point (sink) has flow lines that are the same as the lines of force of the electric field given in a uniform dielectric by equal positive and negative charges at the source and sink. However we do not need to do the complicated Biot-Savart integration over this diffuse current distribution; there is a much simpler approach. In the electrostatic case the electric field is simply the vector sum of the radially-symmetric field of each of the charges (outward from the positive charge, inward to the negative charge); see Fig. 10. Exactly the same is true for our electric current, so the magnetic field produced by the diffuse current distribution is simply the vector sum of the magnetic fields given by the two separate displaced radial current distributions. As is shown by (35), each radial current distribution gives the same magnetic field as that (29) of a semi-infinite line current starting at the surface source/sink. So the magnetic field of the diffuse current between source and sink is exactly the same as that of the two semi-infinite line-currents.

For a horizontal ground surface, and horizontally-stratified ground conductivity, for any one source or sink the resulting magnetic field is everywhere horizontal, and circular about the source/sink; viewed from above it is clockwise for a source pushing current into the ground. In this case, for *any* track geometry and leakage distribution, the magnetic field produced above the ground by the underground part of the leakage current distribution is everywhere horizontal. Therefore close to the surface it is orthogonal to the vertical field produced by the along-track integrated leakage current. Close to a point leak, for example an earthed substation, this horizontal field could be comparable in magnitude to the vertical field produced by the integrated along-track leakage current. But for diffuse leakage, near the track the vertical field will dominate.

So for a given leakage current source $j_L dx$ at position x along the railway we can apply (29) to give the horizontal circular field it produces at an observation point. The total field is given by integrating this over the whole source/sink distribution for that segment of track; we do not need to

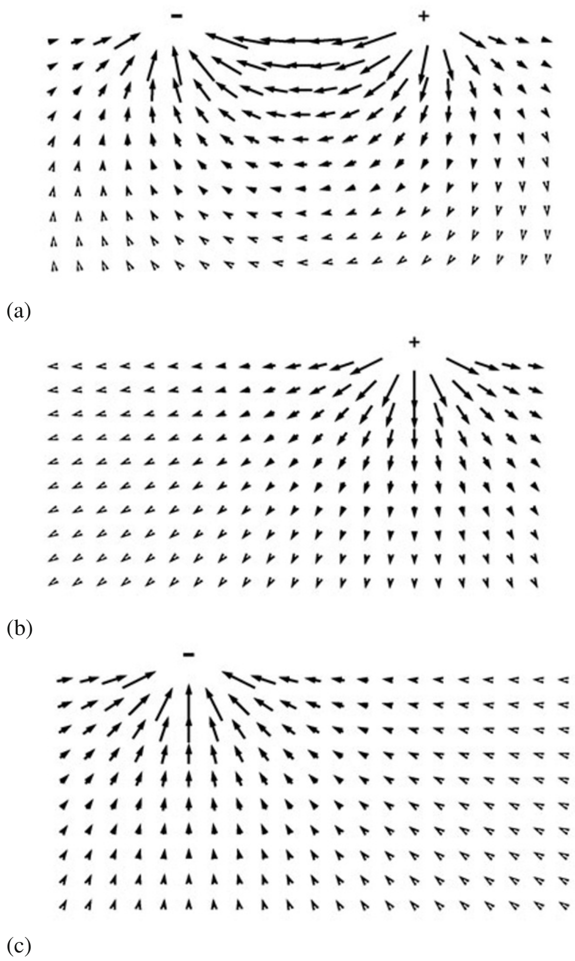

Fig. 10. (a) A 2-dimensional central cross-section through the 3-dimensional current system connecting a point source and a point sink in a uniformly conducting ground having a horizontal top surface. This current system is the vector sum of two radially symmetric current systems, having the same geometry as the electric field from a point charge, (b) for the source, and (c) for the sink. For (b) or (c) the magnetic field produced above the ground is the same as that produced by a semi-infinite vertical line current starting or stopping at the surface.

match a given source with a particular sink. (If a substation has a low-resistance earth, then all the appropriate return current is put at that point.) This is what Dupouy (1950), Tokumoto and Tsunomura (1984), Georgescu *et al.* (2002) and Pirjola *et al.* (2007) did. Again this integration over sources and sinks will involve significant cancellation of the resultant magnetic field.

However, as with the along-rail part of the leakage current, a simpler approximate calculation, based on an estimate of total leakage current, might be all that is justified.

6. Discussion

Railways vary widely in the amplitude and waveform of their train traction current. Before attempting any calculation of magnetic field it is essential to obtain as much information as possible from the railway power-supplies engineer. Given the maximum traction current drawn by one train, and the locations of the substations, for a given train position the current drawn from the nearest substations can then be estimated—see Section 2.3. There is likely to be maximum earth-leakage current when the train is furthest from a substation, but unless you are near an earthed substation the magnetic effect will probably be larger the nearer the train is to the observation point.

Railways vary even more widely in their earth-leakage characteristics! You need to get some feel for the likely magnitude of the diffuse leakage current (e.g. from the along-track resistance ρ, and a *realistic* estimate of the rail-earth leakage conductance σ), and whether the track is earthed at the substations (unlikely). Are there likely to be additional point leaks, particularly if the Depot is not isolated? Even given all the leakage information available to the engineer, it is possible that this will not be sufficient to justify a formal integration (algebraic or numeric) over the track. In this case you might be limited to making a simple approximation of the likely maximum magnetic field produced in normal running conditions. In most cases it is probably sufficient to consider only the nearest train, and only the leakage current in the track segment containing that train.

The algebra given above can be used to calculate the magnetic fields, produced by known traction and leakage currents, at any observation point. However it is unlikely that observatory-quality magnetic observation is feasible near a railway; at relevant distances only the leakage currents will produce significant magnetic field. The along-rail part of this current produces a magnetic field that is circular about the track, so near the surface this field is vertical, while the diffuse ground-return part produces a complicated horizontal field; which field dominates depends on the position of the observer. In urban situations there might be more than one relevant train, and their fields need to be added vectorially. It is probably simplest to work in terms of local cartesian (north, east, downward) components of the magnetic field.

If you are interested in the field at distances comparable with or larger than the length of that track segment, a simple approximation is usually sufficient. For example, even if you think you know the distribution of the leakage current density j_L, you could integrate over all the (positive) sources $j_L dx$ to give the total leakage current J_L; otherwise make the best estimate you can of J_L. Then, for the effect of the along-track leakage current, I suggest using $\frac{1}{2}J_L$ flowing along the whole track segment; alternatively, in a symmetrical situation, use J_L along the central half of the segment. For the effect of the diffuse return current, for track segments that are not too long, I suggest putting the total rail-to-ground J_L at one point central in the region of positive track, and similarly for the ground-to-rail currents. This is what Yanagihara (1977) did, and found that the approximation had trivial effect beyond about 10 km perpendicular to a 20 km long railway.

Near the track, the field from such individual source or sink currents will fall off as $1/R$, but if the observer is at a comparable distance to the source and sink the two fields will overlap and tend to cancel. For large distances a further approximation is possible, by using J_L multiplied by the effective source-to-sink separation to give the line-dipole moment of a vertical semi-infinite line dipole to be put at a central point. We would then use (33), with the angle λ now in the horizontal plane. The field is still everywhere horizontal; for observation points on a line roughly transverse to the centre of the track, if the above-ground along-rail leakage current is from left to right, the field is towards the track. At these larger distances the field falls off as $1/R^2$.

If you do not know the detailed distribution of the leakage current, probably the best you can do is to make an estimate of the total leakage current J_L, and use one of the approximations suggested above. If you wanted to avoid writing your own programs, you could use the existing programs (available from the authors) of Pirjola *et al.* (2007) to estimate the field for any combination of train and observation position. These programs spread the leakage current uniformly along the track, and put all the return current at the substation, but in this situation will probably give as good an answer for remote observation points as the approximations I suggest above.

The semi-infinite vertical line-current calculation for the magnetic effect of the diffuse earth current is formally valid only for horizontally stratified electrical conductivity. If there are local regions of increased conductivity, the current density will be increased in these regions, and decreased in others; the overall effect will usually be to shorten the current paths. So for remote observation points there will tend to be a reduction of the corresponding magnetic field. Similarly, if there is a pipeline (by its nature horizontal, and near the ground surface) buried in the ground, this will locally concentrate any diffuse leakage current. This will tend to keep the underground part of the current closer to the surface, which for remote points would decrease both the horizontal and vertical components of the magnetic field. But if this pipeline crosses the track in a region where the track is itself leaky, it might significantly increase the leakage from the track. And if the observation point happens to be near the pipeline, the field could be significantly increased.

It should also be borne on mind that the unexpected occurrence of a single defect—such as breakage or theft of a bond, or the shorting of a spark gap—can lead to a large local increase in leakage current and hence magnetic interference.

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Appendix A. History

There are many electrical engineering papers that discuss and solve the earth-leakage equations of my Section 4. Here I consider only papers with geomagnetic applications. Most early papers relate only to the effect on observation of telluric current, or to the magnitude of the magnetic noise, and I will not discuss them.

Rössiger (1942) was probably the first to observe the magnetic field perturbation at sufficient resolution to detect individual train movements; this was at Potsdam Observatory, 1.5 km from the railway.

Probably the first person to try to calculate the magnetic field produced by leakage currents was Dupouy (1950), and his paper incorporated most of the results discussed in the present paper. He pointed out that for a homogeneous ground a point source/sink of current would give an axially-symmetric current distribution, so was magnetically equivalent to a semi-infinite vertical line current (Eqs. (35) and (29) of the present paper), and hence that the leakage currents in the ground give only horizontal fields. He also pointed out that (for observation points on the surface) the traction current in the rail and overhead gives only horizontal magnetic field. In effect (though not explicitly) he used the line-dipole approach of (34) for the field from the rail/overhead current loop (though this contribution was negligible at his measurement distance). For a railway of infinite length he developed the exponential model for leakage currents for two simple one substation/one train situations, and showed how the magnetic field from the leakage currents (both source/sink and along rail) could be obtained by numerically integrating along the railway. By comparing theoretical results for various values of the constant α of Section 4 with observations, he deduced a value $\alpha = (9\,\text{km})^{-1}$ for a stretch of the Paris-Orleans railway line. He even persuaded the railway to measure the currents at the two substations, but they were unsuccessful in trying to measure rail-ground voltage!

At Voyeykovo magnetic observatory, 8 km away from a railway having earthed track, signals were observed in both the telluric and magnetic recordings. After a special train was run, these were identified as the signatures of train movements (Mikerina, 1962). Later (Miuchkiurya, 1966) a magnetic compensator was installed, using the telluric potential-gradient at the observatory to drive compensating currents through coils round the magnetometer.

Linington (1974) discussed the effect on archaeological magnetic surveys near railways. He used the infinite-length approximation (30) for the magnetic field given by an assumed uniform along-track net leakage current, for observation distances up to 1 km. However he pointed out that the actual current will vary with train position and acceleration, and that the earth-leakage current density will vary along the track. He showed how to calculate the expression for the ΔF corresponding to a given $(\Delta B_x, \Delta B_y, \Delta B_z)$, and discussed the advantage of using a fixed reference magnetometer.

Yanagihara (1977) used the full exponential solutions for leakage current density, for all sections of a track having more than one substation. He approximated the exponentials by quadratics, used (26) to give the magnetic field for the net above-surface currents, and the semi-infinite vertical line current approach for the underground leakage currents, and produced analytic expressions for the integrated effect of lengths of straight track. For distant points he approximated the continuous distribution of leakage currents by a point source and sink, and used this to calculate the field from a train on a railway line 30 km from Kakioka magnetic observatory. To test the calculations, the railway provided dummy loads at fixed locations.

The work of Yanagihara was extended by Tokumoto and Tsunomura (1984), but using the full exponential solutions, allowing for multiple substations and multiple trains, for a succession of train positions. However they appear to treat the trains as simple resistive loads, so with finite internal resistance of the rectifiers they had to solve sets of simultaneous equations. For this, and for the numerical integration over the railway, they used a powerful meteorological forecast computer to give the small field at Kakioka observatory 15 km away. (Much of their work was comparing their results with those obtained when the rectifiers were assumed to have no internal resistance.) Their method was checked by measuring at an array of observation points out to 5 km from the line, using a model assuming fixed train positions; there was reasonable agreement.

Lowes (1987a, b), like Linington (1974), was close to the railway, about 100 m away. He measured the vertical field component, so was concerned only with the effective net leakage current flowing along the track. In the early morning he could resolve individual train movements, and was able to follow trains up to 5 km away; by monitoring the signal pattern he was able to tell the railway engineers when they had a new point leak.

Georgescu *et al.* (2002) made more calculations for the Paris-Orleans railway, but used a leakage model in which the leakage current density was uniform, and was restricted to the track between the train and the (earthed) substation. Their equations (3) and (4) are presumably based on (29), though there is a mistake in the second term of (4). Their (5) is presumably based on (26), but it is dimensionally and algebraically wrong. Their model of how the traction current varies with time is also unrealistic.

Pirjola *et al.* (2007) showed in detail how to calculate the magnetic field produced by a given current distribution, in effect using (27) separately for overhead and rail (rather than using the line-dipole approach), and the semi-infinite vertical line current approach for the underground leakage currents. Unfortunately they used the unlikely Georgescu *et al.* (2002) distribution of leakage current, though this did allow them to derive analytic expressions for the integrated effects for lengths of straight-line track in terms of the integrated leakage current J_L. (The programs are available from the authors.) For a straight-line track at Calgary, they measured the maximum change in field intensity, along a transverse line out to $R = 5.5$ km, given by a single train. They obtained a reasonable agreement for the fall-off with distance out to about 4 km, but did not know the absolute value of J_L in this case.

Forbriger (2007) found disturbances on vertical broadband seismic sensors coming from variations in the ambient magnetic field affecting a leaf-spring in the sensor. Particularly large disturbances were noticed for a sensor in Stuttgart, caused by the magnetic field produced by the local tram system 160 m away, and he had to use a Helmholtz coil system driven by a 3-axis fluxgate magnetometer to null the magnetic field variation at the seismic sensor. At the time he attributed the magnetic field to the diffuse earth-leakage currents, but now accepts (private communication, 2007) that it was almost certainly coming from the along-track leakage current.

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F. J. Lowes (e-mail: f.j.lowes@ncl.ac.uk)